Functions on Distributive Lattices with the Congruence Substitution Property: Some Problems of Grätzer from 1964

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Let $L$ be a bounded distributive lattice. For $k \geq 1$, let $S_k(L)$ be the lattice of $k$-ary functions on $L$ with the congruence substitution property (Boolean functions); let $S(L)$ be the lattice of all Boolean functions. The lattices that can arise as $S_k(L)$ or $S(L)$ for some bounded distributive lattice $L$ are characterized in terms of their Priestley spaces of prime ideals. For bounded distributive lattices $L$ and $M$, it is shown that $S_1(L) \cong S_1(M)$ implies $S_k(L) \cong S_k(M)$. If $L$ and $M$ are finite, then $S_k(L) \cong S_k(M)$ implies $L \cong M$. Some problems of Grätzer dating to 1964 are thus solved.

1. THE PROBLEM

Let $L$ be a bounded distributive lattice and let $k \geq 1$. A function $f: L^k \to L$ has the congruence substitution property if, for every congruence $\Theta$ of $L$, and all $(a_1, b_1), ..., (a_k, b_k) \in \Theta$, we have $f(a_1, ..., a_k) \Theta f(b_1, ..., b_k)$. The set of all such functions forms a bounded distributive lattice, denoted $S_k(L)$ (also called the lattice of Boolean functions in [3]). Let $S(L)$ be the lattice of all Boolean functions of finite arity (on the variables $x_1, x_2, ...$).

Grätzer has proposed the following problems [3]:

**Problem 1 (Grätzer, 1964).** Let $L$ and $M$ be bounded distributive lattices such that $S_1(L) \cong S_1(M)$. Is $S_k(L)$ necessarily isomorphic to $S_k(M)$?

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Problem 2 (Grätzer, 1964). Characterize those lattices isomorphic to $S_k(L)$ or $S(L)$ for some bounded distributive lattice $L$.

(See also General Lattice Theory [4], Problem II.14.)

We solve both of these problems (Corollary 5.6, Theorem 6.7, and Theorem 6.9).

Grätzer has also proposed the following problem [3]: Given a bounded distributive lattice $L$, find every bounded distributive lattice $M$ such that $S_k(L) \cong S_k(M)$ (or such that $S(L) \cong S(M)$). (In General Lattice Theory [4], Problem II.13, he asks: To what extent do $S(L)$ and $S_k(L)$ determine the structure of $L$?)

We prove that, for a finite distributive lattice $L$, $S_k(L)$ fully determines $L$; but there are infinitely many pairwise nonisomorphic finite distributive lattices $L_1, L_2, ...$ such that $S(L) \cong S(L_n)$ (Theorem 7.1 and Note 7.2).

Along the way, we completely classify the Boolean functions on a bounded distributive lattice $L$ (Theorem 4.7). Our central result is that $S_1(S_k(L))$ is canonically isomorphic to $S_{k+1}(L)$ (Theorem 5.5).

Our proofs rely heavily on Priestley duality for distributive lattices.

2. HISTORICAL BACKGROUND

Functions on a general algebra with the congruence substitution property are the focus of the theory of affine completeness. (See, for instance, [6].)

It is obvious that every lattice polynomial on a bounded distributive lattice has the congruence substitution property, as does every Boolean algebra polynomial on a Boolean lattice. (For instance, $(x \land y) \lor z \in S_1(L)$ if $L$ is Boolean). Grätzer proved the converse ([2], Theorem 1): Every function on a Boolean lattice with the congruence substitution property is a Boolean algebra polynomial (hence the term “Boolean function”). He also characterized those bounded distributive lattices such that every Boolean function is a lattice polynomial ([3], Corollary 3).

The key result for our purposes is the following

**Theorem [3].** Let $L$ be a bounded distributive lattice with least element $0_L$ and greatest element $1_L$. Let $k \geq 1$ and let $2 := \{0_L, 1_L\}$.

For all $f: L^k \to L$, let $\phi_f: 2^k \to L$ be the restriction of $f$ to $2^k$.

1. For all $f, g \in S_k(L)$, $f = g$ if and only if $\phi_f = \phi_g$.

2. Let $\psi: 2^k \to L$. There exists $f \in S_k(L)$ such that $\phi = \phi_f$ if and only if the interval $[\psi(\tilde{a}), \psi(\tilde{a}) \lor \psi(\tilde{b})]$ is a Boolean lattice for all $\tilde{a}, \tilde{b} \in 2^k$ such that $\tilde{a} < \tilde{b}$. 


3. MATHEMATICAL BACKGROUND, TERMINOLOGY, AND NOTATION (A PRIMER ON PRIESTLEY DUALITY)

The central reference is [1].

Let \( L \) be a bounded distributive lattice; let \( 2 := \{0_L, 1_L\} \), where \( 0_L \) is the least element of \( L \) and \( 1_L \) is the greatest element. For \( a, b \in L \), where \( a \leq b \), let \( [a, b]_L \) be the interval \( \{ c \in L \mid a \leq c \leq b \} \). Let \( \text{Con} \) \( L \) be the congruence lattice of \( L \). For \( \% \in \text{Con} \) \( L \) and \( a, b \in L \), we write \( a \% b \) if \( (a, b) \in \% \).

For \( k \geq 1 \), a function \( f: L^k \to L \) has the congruence substitution property if, for all \( \% \in \text{Con} \) \( L \) and all \( a_1, b_1, \ldots, a_k, b_k \in L \), \( a_i \% b_i \) \( (i = 1, \ldots, k) \) implies \( f(a_1, \ldots, a_k) \% f(b_1, \ldots, b_k) \). The (bounded distributive) lattice of all such functions, also called the \( k \)-ary Boolean functions, is denoted \( S_k(L) \).

If we view the members of \( S_k(L) \) as functions depending on the variables \( x_1, \ldots, x_k \), we can take the union

\[
\bigcup_{k=1}^{\infty} S_k(L)
\]
to get the (bounded distributive) lattice \( S(L) \) of all (finitary) Boolean functions.

Let \( P \) be a poset. A down-set of \( P \) is a subset \( U \subseteq P \) such that, for all \( p \in P \) and \( u \in U \), \( p \leq u \) implies that \( p \in U \). The poset of clopen down-sets of an ordered topological space \( P \), partially ordered by inclusion, is a bounded distributive lattice, denoted \( \mathcal{C}(P) \). (Meet is intersection, join is union, \( 0_{\mathcal{C}(P)} \) is \( \emptyset \), and \( 1_{\mathcal{C}(P)} \) is \( P \).)

A Priestley space \( P \) is a compact (partially) ordered topological space such that, for \( p, q \in P \), \( p \leq q \) implies that \( p \notin U \) and \( q \in U \) for some \( U \in \mathcal{C}(P) \). Given a bounded distributive lattice \( L \), the poset \( P(L) \) of prime ideals forms a Priestley space, with the subbasis

\[
\{ \{ I \in P(L) \mid a \in I \}, \{ I \in P(L) \mid a \notin I \} \mid a \in L \}.
\]

It is well known that \( L \) is isomorphic to \( \mathcal{C}(P(L)) \) via the map

\[
a \mapsto U_a := \{ I \in P(L) \mid a \notin I \}.
\]

It is also well known that every Priestley space \( P \) is order-homeomorphic (i.e., order-isomorphic and homeomorphic via the same function) to \( P(\mathcal{C}(P)) \) by the map

\[
p \mapsto I_p := \{ U \in \mathcal{C}(P) \mid p \notin U \}.
\]

Indeed, the category \( \mathbf{D} \) of bounded distributive lattices with \( \{0, 1\} \)-preserving homomorphisms is dually equivalent to the category \( \mathbf{P} \) of Priestley spaces.
with continuous order-preserving maps. [If \( L \) is a finite distributive lattice, and \( \mathcal{J}(L) \) is its poset of join-irreducibles, then \( L \cong \mathcal{J}(\mathcal{J}(L)) \). If \( P \) is a finite poset, then \( P \cong \mathcal{J}(\mathcal{J}(P)) \).]

Under the dual equivalence functor, a map \( f: L \rightarrow M \) in \( D \) corresponds to the map \( \phi: P(M) \rightarrow P(L) \) in \( P \) given by \( \phi(J) = f^{-1}(J) \) for all \( J \in P(M) \).

Similarly, a map \( \varphi: P \rightarrow Q \) in \( P \) corresponds to the map \( f: \mathcal{C}(Q) \rightarrow \mathcal{C}(P) \) in \( D \) given by \( f(V) = \varphi^{-1}(V) \) for all \( V \in \mathcal{C}(Q) \). (See [8]; [1], 10.25.)

If \( L, M \in D \), every prime ideal of \( L \times M \) is of the form \( I \times M \) or \( L \times J \), where \( I \in P(L) \) and \( J \in P(M) \) ([1], Exercise 9.3). If \( M \) is a \([0,1] \)-sublattice of \( L \in D \), then every \( J \in P(M) \) is of the form \( I \cap M \) for some \( I \in P(L) \); moreover, the function \( I \mapsto I \cap M \) is a continuous order-preserving map from \( P(L) \) onto \( P(M) \).

It is well known (Nachbin’s Theorem, [4], Theorem II.1.22) that \( L \in D \) is Boolean if and only if \( P(L) \) is an antichain (that is, distinct elements are incomparable).

In the sequel, let \( P \in P \) and let \( L := \mathcal{C}(P) \).

Every clopen subset of \( P \) is a Priestley space; and for \( U, V \in \mathcal{C}(P) \), \( \mathcal{C}(U \setminus V) \) is isomorphic to \([U \cap V, U]\). Every clopen subset of \( P \in P \) is a finite union of sets of the form \( U \setminus V \), where \( U, V \in \mathcal{C}(P) \).

For all \( Q \subseteq P \), let \( \theta_Q := \{(U,V) \in L^2 | U \cap Q = V \cap Q \} \); if \( Q \) is a singleton \( \{p\} \), we write \( \theta_p \). It is well known that \( \text{Con } L = \{\theta_Q | Q \subseteq P \text{ is closed}\} \) ([1], 10.27).

Given \( U \subseteq P \), let \( [u] := \{p \in P | p < u \text{ for some } u \in U\} \); let \( \text{Max } U \) be the set of maximal elements of the poset \( U \); let \( U^0 := P \setminus U \) and let \( U^1 := U \).

Let \( \mathcal{S}_{L}(L) \) be the family of \( 2^k \)-tuples

\[
\{(U_{\delta})_{\delta \in 2^k} \in L^{2^k} \mid \text{ for all } \delta, \bar{\delta} \in 2^k, \delta < \bar{\delta} \text{ implies } U_{\delta} \subseteq U_{\bar{\delta}}\}.
\]

(Note that \( \mathcal{S}_{L}(L) \) is \([0,1] \)-sublattice of \( L^{2^k} \).

For all \( p \in P, \bar{\delta} \in 2^k \), let

\[
I_{p,\bar{\delta}} := \{(U_{\delta})_{\delta \in 2^k} \in \mathcal{S}_{L}(L) | p \not\in U_{\bar{\delta}}\}.
\]

We know that \( P(\mathcal{S}_{L}(L)) = \{I_{p,\bar{\delta}} | p \in P, \bar{\delta} \in 2^k\} \).

An element \( p \in P \) is normal if there exist \( U, V \in L \) such that \( p \in U, p \not\in V \), and \([U \cap V, U]\) is a Boolean lattice; otherwise \( p \) is special. (Note that, if \( L \) is finite, every \( p \in P \) is normal.)

For any ordered topological space \( R \), let \( P \times R \) be the ordered topological space with underlying space \( P \times R \) and partial ordering

\[
\leq_{P \times R} := \leq_{P \times R} \setminus \{(p, r), (p, r') \in (P \times R)^2 | p \text{ is normal and } r \neq r'\}.
\]
We denote the $i$th component of $\bar{e} \in 2^k$ by $e_i$ ($1 \leq i \leq k$); $\bar{e}'$ denotes the element of $2^{k+1}$ such that
$$
(e')_i = \begin{cases} e_i & \text{if } 1 \leq i \leq k, \\ 0 & \text{if } i = k + 1.
\end{cases}
$$
Similarly, we define $e' \in 2^{k+1}$; $\bar{e}'$ is the complement of $\bar{e}$ in $2^k$.

4. THE LATTICE OF $k$-ARY BOOLEAN FUNCTIONS

In this section, we completely characterize the $k$-ary Boolean functions on a bounded distributive lattice $L$ (Theorem 4.7). In so doing, we obtain Grätzer's result that every $f \in S_k(L)$ is determined by its restriction to $2^k$, where $2 := \{0_L, 1_L\}$; we also obtain a new description of the functions $\phi: 2^k \to L$ that are restrictions of Boolean functions to $2^k$ [easily seen to be equivalent to Grätzer's ([3], Theorem)].

In the sequel, let $P$ be a Priestley space and let $L$ be the bounded distributive lattice $\mathcal{C}(P)$.

We begin with some trivial observations.

**Note 4.1.** Let $U \in \mathcal{C}(P)$. Then $\downarrow U = U \setminus \text{Max } U$.

**Proof.** Every clopen subset of $P$ is in $P$, and so corresponds to the poset of prime ideals of some bounded distributive lattice. By Zorn’s Lemma, every prime ideal in such a lattice is contained in a maximal lattice.

**Lemma 4.2.** Let $U, V, Q \subseteq P$. Then $U \cap Q = V \cap Q$ implies
$$
(P \setminus U) \cap Q = (P \setminus V) \cap Q.
$$

**Note 4.3.** Let $U, V \in \mathcal{C}(P)$. The following are equivalent:

1. $\downarrow U \subseteq V$;
2. $U \setminus V$ is an antichain;
3. $[U \cap V, U]_L$ is a Boolean lattice;
4. $[V, U \cup V]_L$ is a Boolean lattice.

**Proof.** Clearly (1) is equivalent to (2), (2) is equivalent to (3), and (3) is equivalent to (4).
Lemma 4.4. Let \( f \in S_k(L) \). Then for all \( U_1, \ldots, U_k \in L \),
\[
f(U_1, \ldots, U_k) = \bigcup_{i=2^k} \bigcap_{i=1}^{k} f(\bar{i}) \cap U_i^*.
\]

Proof. Let \( p \in P \); let \( U_1, \ldots, U_k \in \mathcal{C}(P) \).
For \( i = 1, \ldots, k \), let
\[
e_i = \begin{cases} 1 & \text{if } p \in U_i, \\ 0 & \text{if } p \notin U_i \\
\end{cases}
\]
(so that \( p \in U_i^* \) and \( U_i \theta_p e_i \)). Hence \( p \in f(U_1, \ldots, U_k) \) if and only if \( p \in f(e_1, \ldots, e_k) \).
Now assume that \( p \in \bigcap_{i=1}^{k} f(\bar{i}) \cap U_i^* \) for some \( \bar{i} \in 2^k \). Then \( U_i \theta_p e_i \) for \( i = 1, \ldots, k \), so that \( f(U_1, \ldots, U_k) = f(\bar{i}) \) and hence \( p \in f(U_1, \ldots, U_k) \). \( \square \)

Lemma 4.5. Let \( f \in S_k(L) \). Then \( (f(\bar{i}))_{\bar{i} \in 2^k} \) is in \( \mathcal{S}_k(L) \).

Proof. Let \( \delta, \bar{i} \in 2^k \) be such that \( \delta < \bar{i} \). Assume for a contradiction that \( f(\bar{i}) \neq f(\bar{i}) \). Then there exist \( p, q \in f(\bar{i}) \) such that \( p < q \) and \( p \neq f(\bar{i}) \).
Let \( U \in \mathcal{C}(P) \) be such that \( p \in U \) and \( q \notin U \). Then \( U \theta_p 1_L \) and \( U \theta_q 0_L \).
For \( i = 1, \ldots, k \), let
\[
U_i := \begin{cases} U & \text{if } \delta_i < e_i, \\ \delta_i & \text{otherwise,} \\
\end{cases}
\]
so that \( U \theta_p e_i \) and \( U \theta_q \delta_i \).
Hence \( q \notin f(U_1, \ldots, U_k) \), so that \( p \notin f(U_1, \ldots, U_k) \); but
\[
p \notin f(U_1, \ldots, U_k),
\]
a contradiction. \( \square \)

Lemma 4.6. Let \( (U_\bar{i})_{\bar{i} \in 2^k} \in \mathcal{S}_k(L) \). Define \( f : L^k \to L \) as follows: for \( U_1, \ldots, U_k \in L \), let
\[
f(U_1, \ldots, U_k) := \bigcup_{\bar{i} \in 2^k} \bigcap_{i=1}^{k} U_\bar{i} \cap U_i^*.
\]
Then \( f \in S_k(L) \) and, for all \( \bar{i} \in 2^k \), \( f(\bar{i}) = U_\bar{i} \).

Proof. First we show that \( f \) is well defined. Let \( U_1, \ldots, U_k \in L \). Clearly \( f(U_1, \ldots, U_k) \) is a clopen subset of \( P \). Let \( p, q \in P \) be such that \( p < q \) where \( q \in f(U_1, \ldots, U_k) \). We must show that \( p \in f(U_1, \ldots, U_k) \).
Assume not, for a contradiction. There exists \( \delta \in 2^k \) such that
\[
g \in \bigcap_{i=1}^{k} U_{x_i} \cap U_{y_i}.
\]
For \( i = 1, ..., k \), let
\[
e_i := \begin{cases} \delta_i & \text{if } p \in U_{y_i}, \\ 1 & \text{otherwise}. \end{cases}
\]
For some \( j \in \{1, ..., k\} \), \( \delta_j = 0 \) and \( \epsilon_j = 1 \) (or else
\[
p \in \bigcap_{i=1}^{k} U_{x_i} \cap U_{y_i},
\]
a contradiction). Hence \( \delta < \epsilon \). Thus \( p \in U_x \); and since
\[
p \in \bigcap_{i=1}^{k} U_{x_i} \cap U_{y_i},
\]
we have \( p \in f(U_1, ..., U_k) \), a contradiction. Hence \( f: L^k \to L \) is well defined.
Clearly \( f \in S_k(L) \). (See Lemma 4.2.)

Finally, let \( \bar{e} \in 2^k \). We will show that \( f(\bar{e}) = U_x \). Certainly \( \epsilon_i = 0 \) for \( i = 1, ..., k \), so
\[
\bigcap_{i=1}^{k} U_{x_i} \cap \epsilon_i = U_x.
\]
Now let \( \bar{\delta} = 2^k \) be distinct from \( \bar{e} \). Then there exists \( i \in \{1, ..., k\} \) such that
\( \delta_i \neq \epsilon_i \). If \( \delta_i = 0 \) and \( \epsilon_i = 1 \), we have \( \epsilon_i = \emptyset \). If \( \delta_i = 1 \) and \( \epsilon_i = 0 \), we have \( \epsilon_i = \emptyset \). Hence
\[
\bigcap_{i=1}^{k} U_{x_i} \cap \epsilon_i = \emptyset.
\]
Thus \( f(\bar{e}) = U_x \). 

The main theorem of this section provides an alternate, unified proof of both \([2]\), Theorem 1 and \([3]\), Theorem. (Note the similarity with \([5]\), Theorem 2.41, which the author came across after proving the main theorem: \([5]\), Theorem 2.41 deals with normal forms for propositional formulas.) Our result extends these theorems by explicitly describing all possible \( k \)-ary Boolean functions.
FIG. 1. The poset $P$ and the lattice $L = \mathfrak{c}(P)$.

**Theorem 4.7.** The lattices $S_k(L)$ and $\mathcal{A}_k(L)$ are isomorphic.

Define a map $\Phi: S_k(L) \rightarrow \mathcal{A}_k(L)$ as follows: for all $f \in S_k(L)$, let

$$\Phi(f) := (f(\emptyset))_{i \in 2^k}.$$

Define a map $\Psi: S_k(L) \rightarrow S_k(L)$ as follows: for all $(U_0)_{i \in 2^k} \in S_k(L)$, let $\Psi((U_0)_{i \in 2^k}) : L^k \rightarrow L$ be the function defined for all $U_1, \ldots, U_k \in L$ by

$$\Psi((U_0)_{i \in 2^k})(U_1, \ldots, U_k) := \bigcup_{i \in 2^k} \bigcap_{i=1}^k U_i \cap U_i^\gamma.$$

Then $\Phi$ and $\Psi$ are mutually inverse order-isomorphisms.

**Proof.** The theorem follows from Lemmas 4.4-4.6.

The theorem implies that the generic unary Boolean function $f: L \rightarrow L$ is given by

$$f(U) = (U_0 \setminus U) \cup (U_1 \cap U),$$

where $U_0, U_1 \in L$ are such that $\downarrow U_0 \subseteq U_1$.

**Example 4.8.** Let $P$ be the two-element chain $\{a, b\}$ where $a < b$; then $L = \mathfrak{c}(P)$ is the three-element chain $\{\emptyset, a, ab\}$ (Fig. 1).

Clearly $\downarrow \emptyset = \downarrow a = \emptyset$ and $\downarrow ab = a$ (Table I).

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\downarrow U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$a$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$ab$</td>
<td>$a$</td>
</tr>
</tbody>
</table>
Hence $\mathcal{S}_1(L)$ is the lattice
\[
\{(\emptyset, \emptyset), (\emptyset, a), (\emptyset, ab), (a, \emptyset), (a, a), (a, ab), (ab, a), (ab, ab)\}
\]
(Fig. 2).

The lattice $\mathcal{S}_2(L)$ has 52 elements, which we list in $2 \times 2$ matrix notation:
\[
\begin{align*}
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\emptyset & \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \\
\end{align*}
\]

Example 4.9. Let $P$ be the three-element fence $\{a, b, c\}$ where $b > a < c$; then $L = \mathcal{E}(P)$ is the lattice $\{\emptyset, a, ab, ac, abc\}$ (Fig. 3).

Clearly $\downarrow \emptyset = \downarrow a = \emptyset$ and $\downarrow ab = \downarrow ac = \downarrow abc = a$ (Table II).

Then $\mathcal{S}_1(L)$ is the lattice $\{\emptyset, a\} \times L \cup \{ab, ac, abc\} \times \{a, ab, ac, abc\}$ (Fig. 4).

Example 4.10. Let $Q$ be the four-element fence $\{w, x, y, z\}$, where $w < x > y < z$; then $M = \mathcal{E}(Q)$ is the lattice $\{\emptyset, w, x, y, wy, wz, wxy, wxyz\}$ (Fig. 5).

Clearly $\downarrow \emptyset = \downarrow w = \downarrow x = \emptyset$, $\downarrow y = \downarrow wz = y$, and $\downarrow wxy = \downarrow wxyz = wy$ (Table III).
FIG. 2. The lattice $\mathcal{Y}(L)$.

FIG. 3. The poset $P$ and the lattice $L = \mathcal{C}(P)$.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\mathcal{C}(U)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$a$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$ab$</td>
<td>$a$</td>
</tr>
<tr>
<td>$ac$</td>
<td>$a$</td>
</tr>
<tr>
<td>$abc$</td>
<td>$a$</td>
</tr>
</tbody>
</table>
FIG. 4. The lattice $\mathcal{L}(L)$.

FIG. 5. The poset $Q$ and the lattice $M = \mathcal{E}(Q)$.
5. BOOLEAN FUNCTIONS ON THE LATTICE OF BOOLEAN FUNCTIONS: THE SOLUTION TO GRÄTZER'S FIRST PROBLEM

In this section, we solve Problem 1 of Section 1, posed by Grätzer in 1964 (Corollary 5.6): The lattice $S_k(L)$ (for a bounded distributive lattice $L$) is determined up to isomorphism by the lattice $S_1(L)$. Indeed, we prove the surprising result that $S_{k+1}(L)$ is canonically isomorphic to $S_1(S_k(L))$, the lattice of unary Boolean functions on the lattice of $k$-ary Boolean functions of $L$ (Theorem 5.5).

Recall that $P \in \mathcal{P}$ and $L = \mathcal{C}(P)$. 

Thus $\mathcal{F}(M)$ has 52 elements:

\[
\begin{align*}
(0,0) & (0,y) (0,yz) (0,wy) (0,wyz) (0,wxy) (0,wyx) \\
(y,0) & (y,y) (y,yz) (y,w) (y,wy) (y,wyz) (y,wxy) (y,wyx) \\
(yz,y) & (yz,yz) (yz,wy) (yz,wyz) (yz,wxy) (yz,wyx) \\
w,0 & (w,y) (w,yz) (w,wy) (w,wyz) (w,wxy) (w,wyx) \\
w,0 & (w,y) (w,yz) (w,wy) (w,wyz) (w,wxy) (w,wyx) \\
(wyz,y) & (wyz,yz) (wyz,wy) (wyz,wyz) (wyz,wxy) (wyz,wyx) \\
(wyz,y) & (wyz,yz) (wyz,wy) (wyz,wyz) (wyz,wxy) (wyz,wyx) \\
(wyz,y) & (wyz,yz) (wyz,wy) (wyz,wyz) (wyz,wxy) (wyz,wyx) \\
\end{align*}
\]
Lemma 5.1. Let $U := (U_\sigma)_{\sigma \in 2^k}$, $\bar{V} := (V_\sigma)_{\sigma \in 2^k} \in \mathcal{G}(L)$ be such that

$$[U_\sigma \cap V_\sigma, U_\sigma]_L$$

is a Boolean lattice whenever $\sigma, \bar{\sigma} \in 2^k$ and $\sigma < \bar{\sigma}$. Choose $W_\sigma \in [U_\sigma \cap V_\sigma, U_\sigma]_L$ for all $\sigma \in 2^k$.

Then $(W_\sigma)_{\sigma \in 2^k}$ belongs to $\mathcal{G}(L)$.

Proof. Let $\sigma, \bar{\sigma} \in 2^k$ such that $\sigma < \bar{\sigma}$. Then

$$\hat{W}_\sigma \equiv \hat{U}_\sigma \equiv U_\sigma \cap V_\sigma \equiv W_\sigma$$

(using Note 4.3).

Corollary 5.2. Let $U := (U_\sigma)_{\sigma \in 2^k}$, $\bar{V} := (V_\sigma)_{\sigma \in 2^k} \in \mathcal{G}(L)$ be such that

$$[U_\sigma \cap V_\sigma, U_\sigma]_L$$

is a Boolean lattice whenever $\sigma, \bar{\sigma} \in 2^k$ and $\sigma \leq \bar{\sigma}$.

Then

$$[\hat{U} \land \bar{V}, \hat{U}]_{\mathcal{G}(L)}$$

is a Boolean lattice.

Proof. Let

$$\hat{W} := (W_\sigma)_{\sigma \in 2^k} \in [\hat{U} \land \bar{V}, \hat{U}]_{\mathcal{G}(L)}.$$ 

Thus, for all $\sigma \in 2^k$, $W_\sigma \in [U_\sigma \cap V_\sigma, U_\sigma]_L$, so there exists $W_\sigma \in [U_\sigma \cap V_\sigma, U_\sigma]_L$ such that $W_\sigma \cap W_\sigma = U_\sigma \cap V_\sigma$ and $W_\sigma \cup W_\sigma = U_\sigma$.

By Lemma 5.1, $W := (W_\sigma)_{\sigma \in 2^k}$ belongs to $\mathcal{G}(L)$; clearly $\hat{W} \land \hat{W} = \hat{U} \land \bar{V}$ and $\hat{W} \lor \hat{W} = \hat{U}$.

Lemma 5.3. Let $U_0 := (U_\sigma)_{\sigma \in 2^k}$, $\bar{U}_1 := (U_\sigma)_{\sigma \in 2^k} \in \mathcal{G}(L)$ be such that $(U_0, \bar{U}_1)$ belongs to $\mathcal{G}(L)$.

Then $[U_0 \equiv \bar{U}_1 \equiv U_0]_L$ for all $\sigma, \bar{\sigma} \in 2^k$ such that $\sigma < \bar{\sigma}$.

Proof. Fix $\sigma, \bar{\sigma} \in 2^k$ such that $\sigma < \bar{\sigma}$. By Note 4.3,

$$[\hat{U}_0 \land \bar{U}_1, \hat{U}_0]_{\mathcal{G}(L)}$$

is a Boolean lattice.
For all $\bar{\eta} \in 2^k$, let

$$W_{\bar{\eta}} := \begin{cases} 
U_{\bar{\eta}} \cap U_{\bar{\eta}_0} & \text{if } \bar{\eta} < \bar{\epsilon} \\
U_{\bar{\eta}_0} & \text{otherwise.}
\end{cases}$$

Then $\bar{W} := (W_{\bar{\eta}})_{\bar{\eta} \in 2^k} \in S_k(L)$; indeed

$$\bar{W} \in [\bar{U}_0 \wedge \bar{U}_1, \bar{U}_0]_{S(L)}.$$  

Let $\bar{W}' := (W_{\bar{\eta}})_{\bar{\eta} \in 2^k} \in S_k(L)$ be such that $\bar{W} \wedge \bar{W}' = \bar{U}_0 \wedge \bar{U}_1$ and $\bar{W} \lor \bar{W}' = \bar{U}_0$.

Clearly $\bar{W}' = U_{\bar{\eta}_1}$ and $\bar{W} = U_{\bar{\eta}_0} \cap U_{\bar{\eta}_1}$. Hence $\downarrow U_{\bar{\eta}_1} \subseteq U_{\bar{\eta}_0}$.  

**Lemma 5.4.** Let $\bar{U}_0 := (U_{\bar{\eta}})_{\bar{\eta} \in 2^k}$, $\bar{U}_1 := (U_{\bar{\eta}})_{\bar{\eta} \in 2^k} \in S_k(L)$ be such that $(\bar{U}_0, \bar{U}_1)$ belongs to $S_k(S_k(L))$.

Then for all $\bar{\epsilon} \in 2^k$, $\downarrow U_{\bar{\epsilon}_1} \subseteq U_{\bar{\epsilon}_0}$.  

**Proof.** Fix $\bar{\epsilon} \in 2^k$. It suffices to prove that $[U_{\bar{\eta}_0} \cap U_{\bar{\eta}_1}, U_{\bar{\eta}_0}]_L$ is a Boolean lattice. Let $W \in [U_{\bar{\eta}_0} \cap U_{\bar{\eta}_1}, U_{\bar{\eta}_0}]_L$.

For all $\bar{\eta} \in 2^k$, let

$$W_{\bar{\eta}} := \begin{cases} 
U_{\bar{\eta}} \cap U_{\bar{\eta}_0} & \text{if } \bar{\eta} < \bar{\epsilon}, \\
W & \text{if } \bar{\eta} = \bar{\epsilon} \\
U_{\bar{\eta}_0} & \text{otherwise.}
\end{cases}$$

Then $\bar{W} := (W_{\bar{\eta}})_{\bar{\eta} \in 2^k} \in S_k(L)$, and it lies in the Boolean interval

$$[\bar{U}_0 \wedge \bar{U}_1, \bar{U}_0]_{S(L)}.$$  

Let $\bar{W}' := (W_{\bar{\eta}})_{\bar{\eta} \in 2^k} \in S_k(L)$ be such that $\bar{W} \wedge \bar{W}' = \bar{U}_0 \wedge \bar{U}_1$ and $\bar{W} \lor \bar{W}' = \bar{U}_0$. Clearly $W \wedge W' = U_{\bar{\epsilon}_1}$ and $W \lor W' = U_{\bar{\epsilon}_0}$.  

**Theorem 5.5.** The lattices $S_{k+1}(L)$ and $S_1(S_k(L))$ are isomorphic.

Define a map

$$\Phi: S_{k+1}(L) \to S_k(S_k(L))$$

as follows: for all $(U_{\bar{\gamma}})_{\bar{\gamma} \in 2^{k+1}} \in S_{k+1}(L)$, let

$$\Phi((U_{\bar{\gamma}})_{\bar{\gamma} \in 2^{k+1}}) = ((U_{\bar{\gamma}})_{\bar{\gamma} \in 2^k}, (U_{\bar{\gamma}})_{\bar{\gamma} \in 2^k}).$$

Define a map

$$\Psi: S_k(S_k(L)) \to S_{k+1}(L)$$
as follows: for all \((U_{2^k})_{k \in 2^k}, (U'_{2^k})_{k \in 2^k}\) ∈ \(\mathcal{S}_1(L)\), let

\[ \Psi((U_{2^k})_{k \in 2^k}, (U'_{2^k})_{k \in 2^k}) = (U'_{2^k})_{k \in 2^k+1}. \]

Then \(\Phi\) and \(\Psi\) are mutually inverse order-isomorphisms.

Proof. By Corollary 5.2 and Note 4.3, \(\Phi\) is well defined. By Lemmas 5.3 and 5.4, \(\Psi\) is well defined. They are clearly order-preserving and inverses to each other.

As a corollary, we solve Grätzer’s first problem ([3]; see Section 1):

**Corollary 5.6.** Let \(L, M \in \mathbf{D}\) be such that \(S_1(L) \cong S_1(M)\).
Then \(S_k(L) \cong S_k(M)\).

**Example 5.7.** Let \(L\) be the three-element chain. In Example 4.8, we computed \(S_1(L)\) and \(S_2(L)\). In Example 4.10, we computed \(S_1(M)\), where \(M \cong S_1(L)\). In both examples, we listed the elements of \(\mathcal{S}_1(L)\) and \(\mathcal{S}_1(S_1(L))\). The isomorphism of Theorem 5.5 can be easily seen by turning each \(2 \times 2\) matrix of Example 4.8 into an ordered pair by grouping the rows together and using the isomorphism \(S_1(L) \cong M\) given by

\[
\begin{align*}
(\emptyset, \emptyset) &\mapsto \emptyset \\
(\emptyset, a) &\mapsto y \\
(\emptyset, ab) &\mapsto yz \\
(a, \emptyset) &\mapsto w \\
(a, a) &\mapsto wy \\
(a, ab) &\mapsto wyz \\
(ab, a) &\mapsto wxy \\
(ab, ab) &\mapsto wxyz.
\end{align*}
\]

6. THE PRIESTLEY DUAL OF THE LATTICE OF BOOLEAN FUNCTIONS: THE SOLUTION TO GRAßTZER’S SECOND PROBLEM

In this section, we solve Problem 2 of Section 1 posed by Grätzer in 1964 and restated in 1978 in his influential book (Theorems 6.7 and 6.9): We completely characterize the lattices that can arise as \(S_k(L)\) or \(S(L)\) for a bounded distributive lattice \(L\). We do so in terms their Priestley spaces of prime ideals.
Recall that $P \in \mathcal{P}$ and $L = \mathcal{C}(P)$.

**Note 6.1.** Let $p \in P$. The following are equivalent:

1. $p$ is normal;
2. there exist $U, V \in \mathcal{C}(P)$ such that $U \setminus V$ is an antichain containing $p$;
3. there exist $W \in \mathcal{C}(P)$ and a clopen subset $C$ of $P$ such that $p \in C \subseteq \operatorname{Max} W$.

**Proof.** Note 4.3 gives the equivalence of (1) and (2) and the fact that (2) implies (3). To show that (3) implies (2), let $U, V \in \mathcal{C}(P)$ be such that $p \in U \setminus V \subseteq C$. Then $U \setminus V$ is an antichain.

**Lemma 6.2.** Let $p, q \in P$ and let $\bar{d}, \bar{e} \in 2^k$. Assume that $p < q$ and $\bar{d} \geq \bar{e}$.

Then $I_{p, \bar{d}} \subseteq I_{q, \bar{e}}$.

**Proof.** Let $(U_{\bar{d}})_{\bar{d} \in 2^k} \in I_{p, \bar{d}}$. Then $p \notin U_{\bar{d}}$. Assume for a contradiction that $q \in U_{\bar{d}}$. Then $p \in \overline{U_{\bar{d}}}$ and hence $p \in U_{\bar{d}}$, a contradiction.

**Lemma 6.3.** Let $p \in P$ and let $\bar{d}, \bar{e} \in 2^k$. Assume that $p$ is special and that $\bar{d} \geq \bar{e}$.

Then $I_{p, \bar{d}} \subseteq I_{p, \bar{e}}$.

**Proof.** Assume for a contradiction that $p \not\leq q$. Let $U \in \mathcal{C}(P)$ be such that $p \notin U$ and $q \in U$. Then $(U_{\bar{d}})_{\bar{d} \in 2^k} \in I_{p, \bar{d}} \setminus I_{q, \bar{e}}$, a contradiction.

**Lemma 6.4.** Let $p, q \in P$ and let $\bar{d}, \bar{e} \in 2^k$. Assume that $I_{p, \bar{d}} \subseteq I_{q, \bar{e}}$.

Then $p \leq q$.

**Proof.** Assume for a contradiction that $p \not\geq q$. Let $U \in \mathcal{C}(P)$ be such that $p \notin U$ and $q \in U$. Then $(U_{\bar{d}})_{\bar{d} \in 2^k} \in I_{p, \bar{d}} \setminus I_{q, \bar{e}}$, a contradiction.

**Lemma 6.5.** Let $p, q \in P$ and let $\bar{d}, \bar{e} \in 2^k$. Assume that $I_{p, \bar{d}} \subseteq I_{q, \bar{e}}$.

Then $\bar{d} \geq \bar{e}$.

**Proof.** Assume for a contradiction that $\bar{d} \not\geq \bar{e}$. For all $\bar{d} \in 2^k$, let

$U_{\bar{d}} := \begin{cases} P & \text{if } \bar{d} \geq \bar{e}, \\ \emptyset & \text{otherwise}. \end{cases}$

Then $(U_{\bar{d}})_{\bar{d} \in 2^k} \in I_{p, \bar{d}} \setminus I_{q, \bar{e}}$, a contradiction.

**Lemma 6.6.** Let $p \in P$ and let $\bar{d}, \bar{e} \in 2^k$. Assume that $I_{p, \bar{d}} \subseteq I_{p, \bar{e}}$ where $\bar{d} \not\geq \bar{e}$.

Then $p$ is special.
Proof. By Lemma 6.5, $\delta \geq \xi$.

Assume, for a contradiction, that $p$ is normal. By Notes 4.3 and 6.1, there exist $U, V \in \mathcal{C}(P)$ such that $p \notin U \setminus V$ and $\Downarrow U \subseteq V$. For all $\eta \in 2^k$, let

$$W_\eta := \begin{cases} V & \text{if } \eta \not\geq \delta, \\ U & \text{if } \eta \not\geq \delta. \end{cases}$$

Then $(W_\eta)_{\eta \in 2^k} \in I_p, \delta \backslash I_p, \xi$, a contradiction. \(\blacksquare\)

**Theorem 6.7.** The Priestley space of $S_k(L)$ is order-homeomorphic to the ordered space $P \times 2^k$.

Define the order-homeomorphism $\Phi: P(\mathcal{J}_k(L)) \to P \times 2^k$ as follows: for all $p \in P$, $\xi \in 2^k$, let $\Phi(I_p, \xi) = (p, \xi)$.

**Proof.** By Lemmas 6.4 and 6.5, $\Phi$ is well defined and order-preserving. By Lemmas 6.2 and 6.3, $\Phi$ is an order-embedding.

Obviously $\Phi$ is onto. Hence $\Phi$ is an order-isomorphism.

To prove that $\Phi$ is a homeomorphism, let

$$\Psi: P(L^{2^k}) \to P(\mathcal{J}_k(L))$$

be the function sending $\{(U_p)_{p \in 2^k} \in L^{2^k} | p \not\in U_p\}$ to $I_p, \xi$ for all $p \in P$, $\xi \in 2^k$.

We know that $\Psi$ is continuous. It is also a bijection. Since Priestley spaces are compact and Hausdorff, $\Psi$ is a homeomorphism (see, for instance, [1], Lemma 10.7A). \(\blacksquare\)

After seeing Theorem 6.7 for finite lattices, M. Maróti made the following observation:

**Corollary 6.8.** If $L$ is finite, then $(\mathcal{J}(S_k(L)), \prec)$ is isomorphic to $(\mathcal{J}(L), \prec) \times (2^k, \leq)$.

**Theorem 6.9.** The Priestley space of $S(L)$ is order-homeomorphic to $P \times 2^\mathbb{N}$.

**Proof.** Clearly $P \times 2^\mathbb{N}$ is a Priestley space. For all $k \in \mathbb{N}$, let

$$\pi_k: P \times 2^\mathbb{N} \to P \times 2^k$$

be the projection onto the $k$th coordinate.
be the obvious projection; similarly, define $\pi_{kl}: P \times 2^l \to P \times 2^k$ for all $k, l \in \mathbb{N}$ such that $k \leq l$.

We verify that $(P \times 2^k, (\pi_{kl}: P \times 2^l \to P \times 2^k)_{k \geq l})$ is the inverse limit of the directed system

$((P \times 2^k)_{k \geq l}, (\pi_{kl})_{1 \leq k < l})$

in the category of Priestley spaces.

**Example 6.10.** Let $P$ be the two-element chain $\{a, b\}$ of Example 4.8 and let $L = \mathcal{C}(P)$ (Fig. 1). Figure 6 shows $P \times 2$ and $P \times 2^2$.

Note that $P \times 2$ is order-isomorphic to $\mathcal{J}(S_1(L))$, so that $\mathcal{C}(P \times 2) \cong S_1(L)$ (Figs. 2 and 7).

Figure 8 shows $P, 2^2, P \times 2$, and $P \times 2^2$.

**Example 6.11.** Let $P$ be the three-element fence $\{a, b, c\}$ of Example 4.9 and let $L = \mathcal{C}(P)$ (Fig. 3). Figure 9 shows $P, P \times 2$, and $P \times 2^2$.

Note that $P \times 2$ is order-isomorphic to $\mathcal{J}(S_1(L))$, so that $\mathcal{C}(P \times 2) \cong S_1(L)$ (Figs. 4 and 10).

Indeed, $\mathcal{J}(S_1(L)) = \{\emptyset, a, (\emptyset, ab), (\emptyset, ac), (a, \emptyset), (ab, \emptyset), (ac, \emptyset)\}$.

**Example 6.12.** Let $Q$ be the four-element fence $\{w, x, y, z\}$ of Example 4.10 and let $M = \mathcal{C}(Q)$ (Fig. 5). Figures 11 and 12 show $Q, Q \times 2$, and $Q \times 2^2$.

Let $P$ be the two-element chain of Example 6.10. Note that $Q \cong P \times 2$ and that $Q \times 2 \cong (P \times 2) \times 2$ is order-isomorphic to $P \times 2^2$ (Fig. 8) under the isomorphism

$$(w, 0) \mapsto (a, x)$$

$$(x, 0) \mapsto (b, x)$$

$$(y, 0) \mapsto (a, 0)$$

$$(z, 0) \mapsto (b, 0)$$

$$(w, 1) \mapsto (a, 1)$$

$$(x, 1) \mapsto (b, 1)$$

$$(y, 1) \mapsto (a, \beta)$$

$$(z, 1) \mapsto (b, \beta).$$
FIG. 6. The posets $P$, $P \times 2$, and $P \Join 2$.

FIG. 7. The lattice $S_1(L)$ and the poset $f(S_1(L))$.

FIG. 8. The posets $P$, $2^1$, $P \times 2^1$, and $P \Join 2^1$.

FIG. 9. The posets $P$, $P \times 2$, and $P \Join 2$. 

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FIG. 10. The lattice $\mathcal{S}_1(L)$ and the poset $\mathcal{J}(\mathcal{S}_1(L))$.

FIG. 11. The posets $Q$ and $Q \times 2$.

FIG. 12. The poset $Q \times 2$. 
7. RECOVERING THE LATTICE FROM THE LATTICE OF BOOLEAN FUNCTIONS

In this section, we address Grätzer’s remaining problem (see Section 1): We prove that a finite distributive lattice $L$ is determined by its lattice of $k$-ary Boolean functions (Theorem 7.1), but not by the lattice of all Boolean functions (Note 7.2).

**Theorem 7.1.** Let $L$, $M$ be finite distributive lattices such that $S_k(L) \cong S_k(M)$. Then $L \cong M$.

**Proof.** Let $P := \mathcal{F}(L)$ and let $Q := \mathcal{F}(M)$. By Theorem 6.7 and Corollary 6.8, $P \times 2^k \cong Q \times 2^k$, so that $(P, <) \times (2^k, \leq) \cong (Q, <) \times (2^k, \leq)$. By [7], Theorem 3, $(P, <) \cong (Q, <)$, so that $P \cong Q$ and hence $L \cong M$.

**Note 7.2.** Let $L$ be a nontrivial finite distributive lattice. Let $\mathcal{M}$ be the family of finite lattices $\{S_k(L) \mid k \geq 1\}$.

Then $S(L) \cong S(M)$ for any $M \in \mathcal{M}$, but no two lattices in $\mathcal{M}$ are isomorphic.

**Proof.** The observation follows from Theorem 7.1 and the fact that, for any $N \in \mathcal{D}$, $S(N)$ is a limit of $\{S_k(N) \mid k \geq 1\}$ in the category $\mathcal{D}$.

**References**