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QUASI-COMPLETENESS AND LOCALIZATIONS OF POLYNOMIAL DOMAINS: A CONJECTURE FROM "OPEN PROBLEMS IN COMMUTATIVE RING THEORY"

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Abstract. It is proved that $k[X_1, \ldots, X_v]$ localized at the ideal $(X_1, \ldots, X_v)$, where k is a field and $X_1, \ldots, X_v$ indeterminates, is not weakly quasi-complete for $v \geq 2$, thus proving a conjecture of D. D. Anderson and solving a problem from "Open Problems in Commutative Ring Theory" by Cahen, Fontana, Frisch, and Glaz.

Our rings are commutative with multiplicative identity. We use terminology from [4] and [5].

Let R be a Noetherian local ring with maximal ideal M. The ring is (*weakly*) *quasi-complete* if, for any decreasing subsequence $\{I_n\}_{n=1}^{\infty}$ of ideals of R (such that $\bigcap_{n=1}^{\infty} I_n = \{0\}$) and any $k \geq 1$, there exists $m \geq 1$ such that $I_m \subseteq \bigcap_{n=1}^{\infty} I_n + M^k$.

In the chapter "Open Problems in Commutative Ring Theory" by Cahen, Fontana, Frisch, and Glaz of the Springer Verlag volume *Commutative Algebra: Recent advances in commutative rings, integer-valued polynomials, and polynomial functions* edited by Fontana, Frisch, and Glaz appears the following.

Problem ([2, Problem 8b]). Let k be a field and let R be the localization of $k[X_1, \ldots, X_v]$ at the ideal generated by the $v \geq 2$ indeterminates $X_1, \ldots, X_v$. Is R (weakly) quasi-complete?

Daniel D. Anderson conjectures that the answer is "no" [1, Conjecture 1] and proves that the answer is "no" if k is countable. His proof depends on the following.

Proposition 1 ([1, Corollary 2, Part 1]). *A Noetherian local integral domain R is weakly quasi-complete if and only if $P \cap R \neq \{0\}$ for each non-zero prime ideal P of $\hat{R}$, the completion of R.*

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Lemma 2 ([1, Example 1])**.** *Let K be a countable field and let $v \geq 2$. Then there exists a non-zero prime ideal P of the ring of formal power series $K[[X_1, \dots, X_v]]$ such that $P \cap K[X_1, \dots, X_v] = \{0\}$.*

We solve the above problem by proving the following.

Theorem 3. *Let k be a field and let $v \geq 2$. Then there exists a non-zero prime ideal Q of $k[[X_1, \dots, X_v]]$ such that $Q \cap k[X_1, \dots, X_v] = \{0\}$.*

Proof. For notational simplicity, we set $v = 2$ and use indeterminates X and Y. Let P be the ideal of the lemma when K is the prime subfield of k. Pick $f \in P \setminus \{0\}$. Let B be a basis of the vector space k over K.

Let $g \in k[[X, Y]]$. For $m, n \geq 0$, the coefficient of $X^m Y^n$ in g is $\sum_{b \in B} z_b^{m,n} b$, where for fixed m and n almost all $z_b^{m,n} \in K$ are 0, and in f it is $a^{m,n} \in K$; in fg it is

$$\sum_{b \in B} \sum_{\substack{r,s,r',s' \geq 0 \\ \text{such that } r+r'=m \text{ and } s+s'=n}} a^{r,s} z_b^{r',s'} b.$$

If $fg \in k[X, Y]$, then there exists $N \geq 0$ such that for all $m, n \geq 0$ with $m + n > N$ we have

$$\sum_{b \in B} \sum_{\substack{r,s,r',s' \geq 0 \\ \text{such that } r+r'=m \text{ and } s+s'=n}} a^{r,s} z_b^{r',s'} b = 0,$$

which means that for all $b \in B$

$$\sum_{\substack{r,s,r',s' \geq 0 \\ \text{such that } r+r'=m \text{ and } s+s'=n}} a^{r,s} z_b^{r',s'} = 0.$$

If $fg \neq 0$, then there exist $\bar{m}, \bar{n} \geq 0$ such that the coefficient of $X^{\bar{m}} Y^{\bar{n}}$ is non-zero, i.e.,

$$\sum_{b \in B} \sum_{\substack{r,s,r',s' \geq 0 \\ \text{such that } r+r'=\bar{m} \text{ and } s+s'=\bar{n}}} a^{r,s} z_b^{r',s'} b \neq 0,$$

so there exists $\bar{b} \in B$ such that

$$\sum_{\substack{r,s,r',s' \geq 0 \\ \text{such that } r+r'=\bar{m} \text{ and } s+s'=\bar{n}}} a^{r,s} z_{\bar{b}}^{r',s'} \neq 0.$$

Letting $\bar{g} \in K[[X, Y]]$ have $z_{\bar{b}}^{m,n}$ as the coefficient of $X^m Y^n$ for $m, n \geq 0$, we see that $f\bar{g}$ is a non-zero element of $K[X, Y]$, so $P \cap K[X, Y] \neq \{0\}$, a contradiction. Thus we have proven.

Claim 1. If $\bar{P}$ is the principal ideal generated by f in $k[[X, Y]]$, then $\bar{P} \cap k[X, Y] = \{0\}$.

Claim 2. The ideal $\bar{P}$ is proper.

Proof. If $1 \in \bar{P}$, then f would be a unit in $k[[X,Y]]$, and hence $a^{0,0} \neq 0$ [5, 1.43]; but then f would be a unit in $K[[X,Y]]$, so P would be improper, a contradiction. □

Since $k[[X,Y]]$ is Noetherian [5, 8.14], by Claim 2 $\bar{P}$ has a primary decomposition $\bar{P} = Q_1 \cap \cdots \cap Q_t$ for some $t \geq 1$ [5, 4.35]. Hence $\sqrt{\bar{P}} = P_1 \cap \cdots \cap P_t$ for prime ideals $P_1, \ldots, P_t$ of $k[[X,Y]]$ [5, 2.30, 4.5].

Claim 3. The intersection $\sqrt{\bar{P}} \cap k[X,Y]$ equals $\{0\}$.

Proof. If there exists a non-zero $g \in k[X,Y]$ such that $g^r \in \bar{P}$ for some $r \geq 1$, then $g^r \in (\bar{P} \cap k[X,Y]) \setminus \{0\}$, contradicting Claim 1. □

Claim 4. For some $i \in \{1, \ldots, t\}$, $P_i \cap k[X,Y] = \{0\}$.

Proof. Assume for a contradiction that for all $i \in \{1, \ldots, t\}$, there exists $g_i \in (P_i \cap k[X,Y]) \setminus \{0\}$. Then $0 \neq g_1 \cdots g_t \in P_1 \cap \cdots \cap P_t \cap k[X,Y] = \sqrt{\bar{P}} \cap k[X,Y]$, contradicting Claim 3. □

Let $Q := P_i$ to prove the theorem. □

Corollary 4. *Let k be a field, R the localization of $k[X_1, \ldots, X_v]$ at the ideal $(X_1, \ldots, X_v)$ where $v \geq 2$. Then R is not weakly quasi-complete.*

References

[1] D. D. Anderson, *Quasi-complete semilocal rings and modules*, In Fontana et al. (eds.), Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and Polynomial Functions, Springer Verlag, New York, 2014.

[2] P.-J. Cahen, M. Fontana, S. Frisch, and S. Glaz, *Open Problems in Commutative Ring Theory*, In Fontana et al. (eds.), Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and Polynomial Functions, Springer Verlag, New York, 2014.

[3] M. Fontana, S. Frisch, and S. Glaz, *Commutative Algebra: Recent Advances in Commutative Rings, Integer-Valued Polynomials, and Polynomial Functions*, Springer Verlag, New York, 2014.

[4] D. G. Northcott, *Ideal Theory*, Cambridge University Press, Cambridge, 1953.

[5] R. Y. Sharp, *Steps in Commutative Algebra*, second edition, Cambridge University Press, Cambridge, 2000.

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