Priestley powers of lattices and their congruences.
A problem of E. T. Schmidt

JONATHAN DAVID FARLEY∗

For Professor E. T. Schmidt on his sixtieth birthday

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Abstract. Let $L$ be a lattice and $M$ a bounded distributive lattice. Let $\text{Con } L$ denote the congruence lattice of $L$, $P(M)$ the Priestley dual space of $M$, and $L^{P(M)}$ the lattice of continuous order-preserving maps from $P(M)$ to $L$ with the discrete topology. It is shown that $\text{Con}(L^{P(M)}) \cong (\text{Con } L)^{P(\text{Con } M)}$, the lattice of continuous order-preserving maps from $P(\text{Con } M)$ to $\text{Con } L$ with the Lawson topology. Various other ways of expressing $\text{Con}(L^{P})$ as a lattice of continuous functions or semilattice homomorphisms are presented. Indeed, a link is established between semilattice homomorphisms from a semilattice $S$ into a bounded distributive lattice $T$ (or its ideal lattice) and continuous order-preserving maps from $P(T)$ into the ideal lattice of $S$ with the Scott, Lawson, or discrete topology. It is also shown that, in general, $\text{Con}(L^{P(M)}) \not\cong (\text{Con } L)^{P(\text{Con } M)}$, solving a problem of E. T. Schmidt (independently solved by Grätzer and Schmidt).

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1. Introduction

A Priestley power of a (semi-) lattice $L$ is a (semi-) lattice $L^P$ of continuous order-preserving maps from a Priestley space $P$ to $L$, where $L$ has the discrete topology (cf. [17], p. 105). (The maps are ordered pointwise.) Every Priestley space arises as the poset of prime filters $P(M)$ of a bounded distributive lattice $M$, appropriately topologized. Hence Boolean powers ([2], Definition IV.5.3) are a special case. If $L$ and $M$ belong to the category $D$ of bounded distributive lattices, then $L^P(M)$ is the coproduct of $L$ and $M$ in $D$ ([5], Corollary 2.3; [6], Theorem and Corollary; [24], Theorem).

In [26], the following problem is stated.

Problem. [26]. If $L$ is a lattice and $M$ a bounded distributive lattice, is the congruence lattice $\text{Con}(L^P(M)) \cong (\text{Con} L)^P(\text{Con} M)$?

The problem has been solved in the affirmative for arbitrary $L$ and finite $M$ ([9], Theorem 2.1) as well as for finite $L$ and arbitrary $M$ ([26], Theorem). We solve the problem completely by showing that

$$\text{Con}(L^P(M)) \cong (\text{Con} L)^P(\text{Con} M),$$

the lattice of continuous order-preserving maps from $P(\text{Con} M)$ to $\text{Con} L$ with the Lawson topology $\Lambda$ (Corollary 6.11). We present an example to show that, in general,

$$\text{Con}(L^P(M)) \ncong (\text{Con} L)^P(\text{Con} M),$$

(Proposition 7.4). Grätzer and Schmidt have proven that the isomorphism holds if and only if either $\text{Con} L$ is finite or $M$ is finite ([15], Theorem 3). Our results were proven independently.

Our approach is to use the results for finite exponents to get the corresponding results for Priestley powers. By [28], Theorem, every Priestley space $P$ is the inverse limit of a filtered system of finite posets $Q$ with the discrete topology. Hence every Priestley power $L^P$ is the filtered limit of lattices $L^Q$. Using an idea of [23], pp. 98–100, we can capture the congruence lattice of such a limit if we know $\text{Con}(L^Q)$ for every $L^Q$ in the system. By [9], Theorem 2.1, we do. (This approach was also taken in [15], §4, but certain non-trivial steps were passed over without proof.)

We represent various types of posets of continuous order-preserving maps as posets of semilattice homomorphisms (Theorem 3.6, Corollaries 3.7 and 3.8). For example, if $S$ is a semilattice with least element $0$ and $T \in D$, then

$$\text{Slat}(S,T) \cong (S^\sigma\Lambda)^P(T),$$
where $S^\sigma$ is the ideal lattice of $S$ and $S^{\sigma\partial}$ this lattice ordered by reverse inclusion.

These representations enable us to provide several alternative representations of $\text{Con}(L^P)$. For example,

$$\text{Comp}(L^P) \cong (\text{Comp } L)^\overline{P}$$

where $L$ is a lattice, $P$ a Priestley space, $\overline{P}$ the same space with the trivial order, and $\text{Comp } L$ the semilattice of compact congruences of $L$ (Theorem 5.10). Also

$$\text{Con}(L^P) \cong (\text{Con } L)^\Sigma,$$

the lattice of continuous maps from $P$ to $\text{Con } L$ where the latter has the Scott topology $\Sigma$ (Theorem 6.7). Alternatively, if $M \in D$, then

$$\text{Con}(L^P(M)) \cong \text{Slat}\left((\text{Comp } L, \lor, 0_{\text{Con } L}), (M_{\text{Bool}}^\sigma, \cap, M_{\text{Bool}})\right),$$

the lattice of semilattice homomorphisms from the $\{0\}$-$\lor$-semilattice $\text{Comp } L$ to the $\{1\}$-$\cap$-semilattice $M_{\text{Bool}}^\sigma$, where $M_{\text{Bool}}$ is the minimal Boolean extension of $M$ (Corollary 6.8). Also

$$\text{Con}(L^P(M)) \cong \text{Slat}\left((\text{Comp } L, \lor, 0_{\text{Con } L}), (\text{Con } M, \cap, 1_{\text{Con } M})\right)$$

(Corollary 6.10). These representations enable us to relate special cases of our results to those of [3] and [14] on semilattice homomorphisms between distributive lattices. In particular, we prove that $\text{Slat}(L, L)$ is self-dual for a finite distributive lattice $L$ (Corollary 3.9). Finally, our representations let us construct the example which yields a negative solution to the problem.

### 2. Notation, definitions, and basic theory

Let us introduce notation and remind ourselves of some definitions and basic results. (See [7], [16], inter alia.) If a poset $P$ has a least element, we denote it $0_P$ or 0; if it has a greatest element, we denote it $1_P$ or 1. A poset with 0 and 1 is bounded.

Denote the ordinal sum of posets $P$ and $Q$ by $P \oplus Q$. Let $\mathcal{P}(X)$ denote the power set of the set $X$. Let $1$ denote the one-one element poset.

Let $P$ be a poset and $Q$ and $S$ subsets. Then $\uparrow_Q S$ denotes

$$\{ p \in Q \mid s \leq p \text{ for some } s \in S \}$$
and \( \downarrow_Q S \) denotes \( \{ p \in Q \mid s \geq p \text{ for some } s \in S \} \).

We also write \( \uparrow S \) for \( \uparrow_P S \) and \( \downarrow S \) for \( \downarrow_P S \). For \( s \in P \), we use \( s \) and \( \downarrow s \) for \( \downarrow \{ s \} \) and \( \downarrow \{ s \} \), respectively. If \( S = \uparrow S \), it is an up-set; if \( S = \downarrow S \), it is a down-set. A non-empty subset \( D \) of \( P \) is directed if every finite subset of \( D \) has an upper bound in \( D \). If \( D \) has a join it is denoted \( \bigcup D \). (The special notation, which is standard, serves as a convenient reminder that the set under consideration is directed.) An ideal is a directed down-set; the set of all such, ordered by inclusion, is denoted \( P^\sigma \). A filtered subset of \( P \) is a directed subset of the poset \( P^\sigma \) whose order is dual to that of \( P \). A filter is an ideal of \( P^\sigma \). The poset of filters of \( P \) is denoted \( P^\sigma \).

An element \( k \) of a poset \( P \) is compact if, for all directed subsets \( D \) of \( P \) such that \( \bigcup D \) exists and \( p \leq \bigcup D \), there exists \( d \in D \) such that \( k \leq d \). The poset of compact elements is denoted \( \kappa(P) \). If \( P \) is a complete lattice, an element \( k \in P \) is compact if and only if, for all \( S \subseteq P \) such that \( k \leq \bigsqcup S \), there exists a finite subset \( T \subseteq S \) such that \( k \leq \bigsqcup T \) ([7], Lemma 3.22). An algebraic lattice is a complete lattice such that every element is a join of compact elements.

The class of semilattices with neutral element is denoted \( \text{Slat} \). [The neutral element is 0 for \( \lor \)-semilattices and 1 for \( \land \)-semilattices ([4], p. 50).] If \( S \) and \( T \in \text{Slat} \), then \( \text{Slat}(S,T) \) denotes the poset of \( \text{Slat} \)-morphisms from \( S \) to \( T \) ordered pointwise, i.e., for \( f, g \in \text{Slat}(S,T) \), \( f \leq g \) if \( f(s) \leq g(s) \) for all \( s \in S \). The subset of \( \text{Slat} \)-morphisms \( f \) whose images \( \text{Im } f \) are finite is denoted \( \text{Slat}^\text{fin}(S,T) \). Let \( \text{Lat} \) be the class of lattices. We regard \( \text{Slat} \) and \( \text{Lat} \) as categories with the appropriate morphisms.

A \( \lor \)-semilattice \( S \) with 0 is distributive if, whenever \( a, x, y \in S \) and

\[
 a \leq x \lor y,
\]

there exist \( b, c \in S \) such that \( b \leq x, c \leq y \), and \( a = b \lor c \). Equivalently, \( S^\sigma \) is a distributive lattice. We shall use Stone duality for the class \( \text{DSlat} \) of distributive \( \lor \)-semilattices with 0 ([13], II.5).

A proper ideal \( I \) of \( S \in \text{DSlat} \) is prime if, whenever \( a, b \in S \) and \( c \leq a, b \) implies \( c \in I \) for all \( c \in S \), then \( a \in I \) or \( b \in I \). For all \( a \in S \), let

\[
 \hat{a} := \{ I \in S^\sigma \mid I \text{ prime, } a \notin I \}.
\]

Let \( S(S) \) be the set of prime ideals of \( S \) with the topology generated by the basis \( \{ \hat{a} \mid a \in S \} \). Then \( S(S) \) is the Stone space of \( S \).

Given a topological space \( X \), let \( \mathcal{O}(X) \) denote the bounded distributive lattice of open sets and \( \mathcal{CO}(X) \) the \( \lor \)-semilattice with least element \( \emptyset \) of compact open
sets. A topological space is **sober** if every non-empty ∪-prime (i.e., ∪-irreducible) closed set is the closure of a point. Stone spaces may be abstractly characterized as sober $T_0$ spaces $X$ such that $\mathcal{CO}(X)$ is a basis. See [29], Lemma 1, Lemma 3, and Satz 4. Indeed, if $S \in \text{DSlat}$, then $\mathcal{CO}\left(S(S)\right) = \{ \hat{a} \mid a \in S \}$. The map $a \mapsto \hat{a}$ $(a \in S)$ is an isomorphism from $S$ onto $\mathcal{CO}\left(S(S)\right)$ ([29], pp. 360–361).

Given $L \in \text{Lat}$, let $\text{Con} L$ denote the lattice of congruences of $L$. It is well-known that $\text{Con} L$ is a distributive algebraic lattice ([1], Theorem II.9.15). For $X \subseteq L \times L$, let

$$\vartheta_L(X) := \bigcap\{ \theta \in \text{Con} L \mid X \subseteq \theta \}.$$  

Let $\text{Comp} L := \kappa(\text{Con} L)$. It is well-known that $\text{Comp} L = \{ \bigvee_{i=1}^n \vartheta_L(a_i, b_i) \mid n \geq 0, a_i, b_i \in L \ (i = 1, \ldots, n) \}$.

If $M \in \text{Lat}$ and $f : L \to M$ is a homomorphism, let

$$\text{Comp}(f) : \text{Comp}(L) \to \text{Comp}(M)$$

denote the function ([23], p. 98)

$$[\text{Comp}(f)](\theta) := \vartheta_M\left(\left(f \times f\right)[\theta]\right) \quad (\theta \in \text{Comp} L).$$

If $A$ is an algebraic lattice, then $\left(\kappa(A), \vee, 0_A\right) \in \text{Slat}$ and $\left(\kappa(A)\right)^\sigma \cong A$ via the map $I \mapsto \bigsqcup I$ $(I \in \kappa(A)^\sigma)$ with inverse $a \mapsto \downarrow_{\kappa(A)} a$ $(a \in A)$. Further, if $(S, \vee, 0_S) \in \text{Slat}$, then $S^\sigma$ is an algebraic lattice and $\kappa(S^\sigma) = \{ \uparrow s \mid s \in S \}$, which is isomorphic to $S$. (See [10], Corollary 2.) Similarly, if $S$ is a bounded lattice then $\kappa(S^\pi) = \{ \downarrow s \mid s \in S \}$. If $A$ is an algebraic lattice, the **Scott topology** is the topology

$$\Sigma := \{ U \subseteq A \mid U = \uparrow U \text{ and for all directed } D \subseteq A, \bigcup D \in U \implies D \cap U \neq \emptyset \}.$$  

The **Lawson topology** is the topology $\Lambda$ on $A$ generated by the subbasis

$$\Sigma \cup \{ A \setminus \uparrow a \mid a \in A \}.$$  

(See [12], pp. 99, 144.)
If $P$ and $Q$ are ordered spaces and $Q$ has topology $\tau$, $Q^P_\tau$ is the poset of continuous order-preserving maps from $P$ to $Q$ ordered pointwise; $Q^P$ is $Q^P_\tau$ where $\tau$ is the discrete topology.

An ordered space $P$ is totally order-disconnected if, for all $p, q \in P$ such that $p \not\leq q$, there exists a clopen up-set $U \subseteq P$ such that $p \in U, q \notin U$. A Priestley space is a compact totally order-disconnected ordered space. Let $P$ denote the category of Priestley spaces with continuous order-preserving maps. Let $P^\text{fin}$ denote the full subcategory of finite Priestley spaces. By the proofs of [12], Theorems III.1.9 and III.1.10, an algebraic lattice with the Lawson topology is a Priestley space.

If $P$ is an ordered space, let $D(P)$ denote the set of clopen up-sets of $P$; let $U(P)$ denote the set of open up-sets.

Let $D$ denote the category of bounded distributive lattices with $\{0, 1\}$-homomorphisms (homomorphisms preserving 0 and 1). For $L \in D$, let $P(L)$ denote the Priestley space of prime filters of $L$, appropriately topologized. Let $J(L)$ denote the poset of join-irreducible elements of $L$. For $a \in L$, let $\rho_L(a) := \{ F \in P(L) \mid a \in F \}$.

It is well-known that $D$ and $P$ are dually equivalent categories, $D(-)$ and $P(-)$ being the functors yielding the dual equivalence. We shall identify a lattice with the clopen up-sets of its Priestley dual space and shall not differentiate between the abstract and concrete forms of the lattice. For the details of Priestley duality, see [20], [21].

If $L \in D$, there is an isomorphism from $L^\sigma$ to $U(P(L))$. Refer to [22], §8; see also [7], 10.24.

If $P \in P$, let $P$ denote the trivially ordered Priestley space with the same topology as $P$. We denote the minimal Boolean extension of $L \in D$ by $L_{\text{Bool}}$. See [1], Definition V.4.5, [21], §6.

If $L \in D$, then $\text{Con} L$ is dually isomorphic to the lattice of closed subsets of $P(L)$ ([7], 10.27).

Let $P$ be a set, $\Pi, \Pi_0$ partitions of $P$. Let $\nu_\Pi: P \to \Pi$ be the map assigning each element of $P$ its equivalence class. The set of partitions of $P$ is ordered as follows: $\Pi \leq \Pi_0$ if every equivalence class of $\Pi$ is contained in some equivalence class of $\Pi_0$. If $\Pi \leq \Pi_0$, let

$$\nu_{\Pi, \Pi_0}: \Pi \to \Pi_0$$

be the map assigning each equivalence class of $\Pi$ the unique equivalence class of $\Pi_0$ containing it.

Given a partition $\Pi := \{V_i\}_{i \in I}$ of a poset $P$ into equivalence classes indexed by a set $I$, we define a quasiorder $\leq_{\Pi}$ on $\Pi$ as follows. Let $\leq_{\Pi}$ be the transitive...
closure of the relation \( \preceq_\Pi \) defined in this way: \( V_i \preceq_\Pi V_j \) if \( p \leq q \) for some \( p \in V_i \), \( q \in V_j \) \((i, j \in I)\).

If \( P \in \mathbf{P} \), denote by \( \mathcal{E}_P \) the ordered set of partitions \( \Pi \) of \( P \) into open equivalence classes such that \( (\Pi, \preceq_\Pi) \) is partially ordered. Regard \( \Pi \) as a space with the discrete topology. The same partition \( \Pi \) with the antichain ordering is denoted \( \Pi^\uparrow \).

Let \( P \in \mathbf{P}, M \in \mathbf{Lat} \cup \mathbf{Slat} \). For every \( \Pi \in \mathcal{E}_P \), let \( \mu^M_\Pi : M^\Pi \to M^P \) be defined by \( \mu^M_\Pi(f) := f \circ \nu_\Pi \) \((f \in M^\Pi)\).

For \( \Pi, \Pi_0 \in \mathcal{E}_P \) such that \( \Pi \preceq_\Pi \Pi_0 \), let

\[
\mu^M_{\Pi, \Pi_0} : M^{\Pi_0} \to M^\Pi
\]

be defined by

\[
\mu^M_{\Pi, \Pi_0}(f) := f \circ \nu_{\Pi, \Pi_0} \quad (f \in M^{\Pi_0}).
\]

For \( L \in \mathbf{Lat}, P \in \mathbf{P}^{\text{fin}}, \) and \( p \in P \), denote by \( \chi_p \) the kernel of the \( p \)-th projection of \( L^P \) onto \( L \). Define

\[
\Gamma'_P : \text{Con}(L^P) \to (\text{Con} L)^\mathcal{P}
\]
as follows: for \( \theta \in \text{Con}(L^P) \) and \( p \in P \), let

\[
[\Gamma'_P(\theta)](p) := \{(a, b) \in L \times L \mid (f, g) \in \theta \lor \chi_p \text{ for all } f, g \in L^P \\
\text{such that } f(p) = a, g(p) = b \}.
\]

Define

\[
\Gamma_P : \text{Comp}(L^P) \to (\text{Comp} L)^\mathcal{P}
\]
by \( \Gamma_P(\theta) := \Gamma'_P(\theta) \) for all \( \theta \in \text{Comp}(L^P) \). Define

\[
\Delta'_P : (\text{Con} L)^\mathcal{P} \to \text{Con}(L^P)
\]
as follows: for \( F \in (\text{Con} L)^\mathcal{P} \), let

\[
\Delta'_P(F) := \{(f, g) \in L^P \times L^P \mid (f(p), g(p)) \in F(p) \text{ for all } p \in P \}.
\]

Define

\[
\Delta_P : (\text{Comp} L)^\mathcal{P} \to \text{Comp}(L^P)
\]
by \( \Delta_P(F) := \Delta'_P(F) \) for all \( F \in (\text{Comp} L)^\mathcal{P} \). (That the above functions are well defined will be shown in Proposition 5.7.)
If \( L \in \text{Lat} \) and \( P \in \text{P}^{\text{fin}}, a, b \in L, \) and \( p_0 \in P, \) define \( m_p(a, b, p_0): P \rightarrow L \) for all \( p \in P \) as follows:

\[
[m_p(a, b, p_0)](p) := \begin{cases} 
  a & \text{if } p = p_0, \\
  a \lor b & \text{if } p > p_0, \\
  a \land b & \text{else.}
\end{cases}
\]

Finally, we remind ourselves of basic categorical notions. Let \( \mathbf{C} \) be a category and \( F \) a filtered poset. Let \((C_i)_{i \in F}\) be a family of objects of \( \mathbf{C} \) and

\[
(f_{ij}: C_j \rightarrow C_i)_{i,j \in F}
\]
a family of morphisms with the following properties:

(1) \( f_{ii} = \text{id}(C_i) \) for all \( i \in F; \)
(2) \( f_{ij} \circ f_{jk} = f_{ik} \) for all \( i, j, k \in F \) such that \( i \leq j \leq k. \)

Then \( \mathcal{S} := (\langle C_i \rangle_{i \in F}, \langle f_{ij} \rangle_{i,j \leq j} \rangle \) is a filtered system in \( \mathbf{C} \). Assume \( C \in \mathbf{C} \) and \((f_i:C_i \rightarrow C)_{i \in F}\) is a family of morphisms such that \( i \leq j \) implies \( f_i \circ f_{ij} = f_j \) \((i, j \in F). \) Then

\[
\left( C, (f_i:C_i \rightarrow C)_{i \in F} \right)
\]
is compatible with the filtered system \( \mathcal{S}. \) Assume \( \left( C, (f_i:C_i \rightarrow C)_{i \in F} \right) \) also has the property that, for any \( \left( C', (f'_i:C_i \rightarrow C')_{i \in F} \right) \) compatible with \( \mathcal{S}, \) there is a unique morphism \( f:C \rightarrow C' \) such that \( f \circ f_i = f'_i \) for all \( i \in F. \) Then \( \left( C, (f_i:C_i \rightarrow C)_{i \in F} \right) \) is a filtered limit of \( \mathcal{S}. \)

Let \((D_i)_{i \in F}\) be a family of objects of \( \mathbf{C} \) and

\[
(g_{ij}: D_i \rightarrow D_j)_{i,j \leq j} \]
a family of morphisms with the following properties:

(1) \( g_{ii} = \text{id}(D_i) \) for all \( i \in F; \)
(2) \( g_{jk} \circ g_{ij} = g_{ik} \) for all \( i, j, k \in F \) such that \( i \leq j \leq k. \)

Then \( \mathcal{T} := \langle (D_i)_{i \in F}, (g_{ij}: D_i \rightarrow D_j)_{i,j \leq j} \rangle \) is an inverse system in \( \mathbf{C}. \) Assume \( D \in \mathbf{C} \) and \((g_i:D \rightarrow D_i)_{i \in F}\) is a family of morphisms such that \( i \leq j \) implies \( g_{ij} \circ g_i = g_j \) \((i, j \in F). \) Then

\[
\left( D, (g_i:D \rightarrow D_i)_{i \in F} \right)
\]
is compatible with the inverse system $T$. Assume $\left( D, (g_i : D \to D_i)_{i \in F} \right)$ also has the property that, for any $(D', (g'_i : D' \to D'_i)_{i \in F})$ compatible with $T$, there is a unique morphism $g : D' \to D$ such that $g_i \circ g = g'_i$ for all $i \in F$. Then $\left( D, (g_i : D \to D_i)_{i \in F} \right)$ is an inverse limit of $T$.

A result will be referred to without a section number in the section in which it appears.

3. Continuous function duals of semilattice homomorphisms

In this section we show how various posets of Slat-morphisms may be viewed as posets of continuous order-preserving maps from a Priestley space into an ideal lattice with an appropriate topology (Theorem 6, Corollary 7, and Corollary 8). We then show how Priestley relations, introduced in [3] as the duals of $\{0\}$-$\lor$-homomorphisms between bounded distributive lattices under Priestley duality, correspond naturally with such function spaces (Proposition 12).

Lemma 3.1. Let $A$ be an algebraic lattice. The family $\{ \uparrow k \mid k \in \kappa(A) \}$ is closed under finite (including empty) intersections and is a basis for $\Sigma$.

Hence $\{ \uparrow k \mid k \in \kappa(A) \} \cup \{ A \uparrow a \mid a \in A \}$ is a subbasis for $\Lambda$. 

Proof. See [12], Corollary II.1.15.

Lemma 3.2. Let $A$ be an algebraic lattice and let $P \in \mathcal{P}$ and $p \in P$. Let 

$$g \in \text{Slat} \left( \left( \kappa(A), \lor, 0_A \right), \left( \mathcal{U}(P), \cap, P \right) \right).$$

Then $\{ k \in \kappa(A) \mid p \in g(k) \} \in \kappa(A)^{\sigma}$. Hence for all $k_0 \in \kappa(A)$,

$$k_0 \leq \bigsqcup \{ k \in \kappa(A) \mid p \in g(k) \} \iff p \in g(k_0).$$

Proof. Let $I := \{ k \in \kappa(A) \mid p \in g(k) \}$. As $g(0_A) = P$, we have $0_A \in I$.

If $k_0 \in \kappa(A)$, $k \in I$, and $k_0 \leq k$, then $p \in g(k) \subseteq g(k_0)$, so $k_0 \in I$.

If $k_0, k_1 \in I$, then $p \in g(k_0) \cap g(k_1) = g(k_0 \lor k_1)$, so $k_0 \lor k_1 \in I$. Therefore $I \in \kappa(A)^{\sigma}$.

By the isomorphism between $\kappa(A)^{\sigma}$ and $A$ of §2, for all $k \in \kappa(A)$, $k \leq \bigsqcup I$ if and only if $k \in I$. 

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Lemma 3.3. Let $A$ be an algebraic lattice and let $P \in \mathbf{P}$. Let $f: P \rightarrow A$ be a map. Assume
\[ \{ f^{-1}(\uparrow k) \mid k \in \kappa(A) \} \]
is finite. Then for all $a \in A$, there exists $k \in \downarrow \kappa(A) a$ such that
\[ f^{-1}(\uparrow a) = f^{-1}(\uparrow k). \]

Proof. Let $a \in A$. Let $k_0 \in \downarrow \kappa(A) a$ be such that $f^{-1}(\uparrow k_0)$ is minimal in
\[ \{ f^{-1}(\uparrow k) \mid k \in \downarrow \kappa(A) a \}. \]
Then for all $k \in \kappa(A)$ such that $k_0 \leq k \leq a$, we have $f^{-1}(\uparrow k) = f^{-1}(\uparrow k_0)$. Therefore
\[ f^{-1}(\uparrow a) = f^{-1} \left( \bigcap_{k \in \kappa(A) \cap \downarrow a} \uparrow k \right) = \bigcap_{k \in \kappa(A) \cap \downarrow a} f^{-1}(\uparrow k) \]
\[ = \bigcap_{\substack{k \in \kappa(A) \cap \downarrow a \atop k_0 \leq k \leq a}} f^{-1}(\uparrow k) = f^{-1}(\uparrow k_0). \]

Lemma 3.4. Let $A$ be an algebraic lattice and let $P \in \mathbf{P}$. Let $f \in A^P$. The following are equivalent:
1. $f \in A^P$;
2. for all $k \in \kappa(A)$, $f^{-1}(\uparrow k)$ is closed.
In either case, for all $k \in \kappa(A)$, $f^{-1}(\uparrow k) \in D(P)$.

Proof. See [16], §V.

Lemma 3.5. Let $A$ be an algebraic lattice and let $P \in \mathbf{P}$. Let $f \in A^P$. The following are equivalent:
1. $f \in A^P$;
2. $\text{Im } f$ is finite;
3. $\{ f^{-1}(\uparrow k) \mid k \in \kappa(A) \}$ is finite.
Proof. (1)⇒(2). The implication holds because $P$ is compact and $f$ is a continuous map into a space with the discrete topology, so $\text{Im} f$ is compact and hence finite.

(2)⇒(3). For all $k \in \kappa(A)$,

$$f^{-1}(\uparrow k) = \bigcup \{ f^{-1}(a) \mid a \in \text{Im} f \text{ and } k \leq a \}$$

so there are at most $2^n$ elements in $\{ f^{-1}(\uparrow k) \mid k \in \kappa(A) \}$ where $n$ is the size of $\text{Im} f$.

(3)⇒(1). By Lemmas 3 and 4, $\{ f^{-1}(\uparrow a) \mid a \in A \}$ is finite and $f^{-1}(\uparrow a)$ is clopen for all $a \in A$. Let $a \in A$. Then

$$\{ f^{-1}(\uparrow b) \mid b \in A \text{ and } a < b \} = \{ f^{-1}(\uparrow b_i) \mid i = 1, \ldots, n \}$$

for some $n \geq 0$, $b_i \in A$ such that $a < b_i$ ($i = 1, \ldots, n$). Then

$$f^{-1}(a) = f^{-1}(\uparrow a) \setminus \left( \bigcup \{ f^{-1}(\uparrow b) \mid b \in A \text{ where } a < b \} \right)$$

$$= f^{-1}(\uparrow a) \setminus \bigcup_{i=1}^{n} f^{-1}(\uparrow b_i),$$

which is open. Hence $f \in A^P$.

Theorem 3.6. Let $A$ be an algebraic lattice and let $P \in \mathbf{P}$. By $\kappa(A)$ and $U(P)$ we shall mean the objects \( \left( \kappa(A), \lor, 0_A \right) \) and \( \left( U(P), \cap, P \right) \) of $\text{Slat}$. Define a map

$$\Psi: A^P_\Sigma \rightarrow \text{Slat} \left( \kappa(A), U(P) \right)$$

as follows: for $f \in A^P_\Sigma$ and $k \in \kappa(A)$, let

$$[\Psi(f)](k) := f^{-1}(\uparrow k).$$

Define a map

$$\Phi: \text{Slat} \left( \kappa(A), U(P) \right) \rightarrow A^P_\Sigma$$

as follows: for $g \in \text{Slat} \left( \kappa(A), U(P) \right)$ and $p \in P$, let

$$[\Phi(g)](p) := \bigsqcup \{ k \in \kappa(A) \mid p \in g(k) \}.$$

Then $\Psi$ and $\Phi$ are mutually-inverse order-isomorphisms. The restriction of $\Psi$ to $A^P_\Lambda$ maps onto $\text{Slat} \left( \kappa(A), D(P) \right)$. The restriction of $\Psi$ to $A^P$ maps onto $\text{Slat}^{\text{fin}} \left( \kappa(A), D(P) \right)$.
We conclude that the map $p$ is continuous from $P$ to $A$. As $f$ is continuous and order-preserving, by Lemma 1 $f^{-1}(\uparrow k) \in \mathcal{U}(P)$. Also $f^{-1}(\uparrow 0_A) = f^{-1}(A) = P$.

Finally $f^{-1}(\uparrow (k_1 \vee k_2)) = f^{-1}(\uparrow k_1) \cap f^{-1}(\uparrow k_2)$. So the map $k \mapsto f^{-1}(\uparrow k) \{ k \in \kappa(A) \}$ is in $\text{Slat}(\kappa(A), \mathcal{U}(P))$. Thus $\Psi$ is well-defined.

Let $f_1, f_2 \in A^P_\kappa$ be such that $f_1 \leq f_2$. For $k \in \kappa(A)$,

$$[\Psi(f_1)](k) = f_1^{-1}(\uparrow k) = \{ p \in P \mid k \leq f_1(p) \} \subseteq \{ p \in P \mid k \leq f_2(p) \} = f_2^{-1}(\uparrow k) = [\Psi(f_2)](k).$$

Hence $\Psi(f_1) \leq \Psi(f_2)$, so $\Psi$ is order-preserving.

Let $g \in \text{Slat}(\kappa(A), \mathcal{U}(P))$ and $p_0 \in P$. By Lemma 2,

$$\{ k \in \kappa(A) \mid p_0 \in g(k) \}$$

is directed. Let $k_0 \in \kappa(A)$ be such that $\bigsqcup \{ k \in \kappa(A) \mid p_0 \in g(k) \} \in \uparrow k_0$. By Lemma 2, $p_0 \in g(k_0)$. As $g(k_0)$ is open, we conclude that the map

$$p \mapsto \bigsqcup \{ k \in \kappa(A) \mid p \in g(k) \} \quad (p \in P)$$

is continuous from $P$ to $A$ with the Scott topology, by Lemma 1.

Let $p_1, p_2 \in P$ be such that $p_1 \leq p_2$. Let $k_0 \in \kappa(A)$ be such that $p_1 \in g(k_0)$. Then $p_2 \in g(k_0)$, because $g(k_0)$ is an up-set. Therefore

$$\bigsqcup \{ k \in \kappa(A) \mid p_1 \in g(k) \} \subseteq \bigsqcup \{ k \in \kappa(A) \mid p_2 \in g(k) \}.$$ 

We conclude that the map $p \mapsto \bigsqcup \{ k \in \kappa(A) \mid p \in g(k) \}$ is order-preserving. Therefore $\Phi$ is well-defined.

Let $g_1, g_2 \in \text{Slat}(\kappa(A), \mathcal{U}(P))$ be such that $g_1 \leq g_2$ and let $p \in P$. For $k \in \kappa(A)$, $p \in g_1(k)$ implies $p \in g_2(k)$, so

$$[\Phi(g_1)](p) = \bigsqcup \{ k \in \kappa(A) \mid p \in g_1(k) \} \subseteq \bigsqcup \{ k \in \kappa(A) \mid p \in g_2(k) \} = [\Phi(g_2)](p).$$

Hence $\Phi(g_1) \leq \Phi(g_2)$, so $\Phi$ is order-preserving.

Let $f \in A^P_\kappa$. For $p_0 \in P$,

$$[(\Phi \circ \Psi)(f)](p_0) = \bigsqcup \{ k \in \kappa(A) \mid p_0 \in [\Psi(f)](k) \} \subseteq \bigsqcup \{ k \in \kappa(A) \mid p_0 \in f^{-1}(\uparrow k) \} = \bigsqcup \{ k \in \kappa(A) \mid k \leq f(p_0) \} = f(p_0),$$

where $f$ is image of $p_0$.
so \((\Phi \circ \Psi)(f) = f\). That is, \(\Phi \circ \Psi = \text{id}(A^\kappa_P)\).

Now let \(g \in \text{Slat}(\kappa(A), \mathcal{U}(P))\). For \(k_0 \in \kappa(A)\),

\[
\begin{align*}
[(\Psi \circ \Phi)(g)](k_0) &= [\Phi(g)]^{-1}(\uparrow k_0) = \{ p \in P \mid k_0 \leq [\Phi(g)](p) \} \\
&= \{ p \in P \mid k_0 \leq \bigcup \{ k \in \kappa(A) \mid p \in g(k) \} \}.
\end{align*}
\]

By Lemma 2, we have

\[
[(\Psi \circ \Phi)(g)](k_0) = \{ p \in P \mid p \in g(k_0) \} = g(k_0),
\]

so \((\Psi \circ \Phi)(g) = g\). That is, \(\Psi \circ \Phi = \text{id}[\text{Slat}(\kappa(A), \mathcal{U}(P))]\). Therefore, \(\Psi\) and \(\Phi\) are mutually-inverse order-isomorphisms.

Let \(f \in A^\kappa_P\). By Lemma 4, \(f \in A^\kappa_P\) if and only if

\[
\Psi(f) \in \text{Slat}(\kappa(A), D(P)).
\]

By Lemma 5, \(f \in A^P\) if and only if \(\text{Im } \Psi(f)\) is finite and

\[
\Psi(f) \in \text{Slat}(\kappa(A), D(P)).
\]

The next corollary follows from Theorem 6 using the \(\mathbf{D-P}\) dictionary for ideals mentioned in §2. It explains the “curious duality” behind the representation of modular lattices of the form \(M^P_3\), where \(M_3\) is the five-element non-distributive modular lattice and \(P\) a finite poset ([25], §1, Construction 1).

**Corollary 3.7.** Let \((S, \lor, 0_S) \in \text{Slat}, T \in \mathbf{D}\). We regard \(T\) and \(T^\sigma\) as the objects \((T, \land, 1_T)\) and \((T^\sigma, \cap, T)\) of \text{Slat}, respectively. Let \(\varphi: \kappa(T^\sigma) \cong T\) be the isomorphism \(\varphi(\downarrow t) = t (t \in T)\). Define a map

\[
\Psi: (S^\sigma_S)^{P(T)} \to \text{Slat}(S, T^\sigma)
\]

as follows: for \(f \in (S^\sigma_S)^{P(T)}\) and \(s \in S\), let

\[
[\Psi(f)](s) := \{ t \in T \mid s \in \bigcap f[\rho_T(t)] \}.
\]

Define a map

\[
\Phi: \text{Slat}(S, T^\sigma) \to (S^\sigma_S)^{P(T)}
\]
as follows: for \( g \in \text{Slat}(S, T^\sigma) \) and \( F \in P(T) \), let
\[ [\Phi(g)](F) := \{ s \in S \mid F \cap g(s) \neq \emptyset \}. \]

Then \( \Psi \) and \( \Phi \) are mutually-inverse order-isomorphisms.

Define \( \Psi' : (S^\sigma_\Lambda)^{P(T)} \to \text{Slat}(S, T) \) as follows: for \( f \in (S^\sigma_\Lambda)^{P(T)} \) and \( s \in S \), let
\[ [\Psi'(f)](s) := \varphi\left(\left(\Psi(f)\right)(s)\right). \]

Define \( \Phi' : \text{Slat}(S, T) \to (S^\sigma_\Lambda)^{P(T)} \) as follows: for \( g \in \text{Slat}(S, T) \) and \( F \in P(T) \), let
\[ [\Phi'(g)](F) := g^{-1}(F). \]

Then \( \Psi' \) and \( \Phi' \) are mutually-inverse order-isomorphisms. The restriction of \( \Psi' \) to \((S^\sigma)^{P(T)}\) maps onto \( \text{Slat}^{\text{fin}}(S, T) \).

By reversing the order of \( T \), we get the following.

**Corollary 3.8.** Let \((S, \vee, 0_S) \in \text{Slat}, \ T \in \mathbf{D}\). We regard \( T \) as the object \((T, \vee, 0_T)\) of \( \text{Slat} \). Let \( \varphi : \kappa(T^\sigma) \to T \) be the dual-isomorphism \( \varphi(\uparrow t) = t \).

Define the map \( \Psi' : (S^\sigma_\Lambda)^{P(T)} \to \text{Slat}(S, T) \) as follows: for \( f \in (S^\sigma_\Lambda)^{P(T)} \) and \( s \in S \), let
\[ [\Psi'(f)](s) := \varphi\left(\{ t \in T \mid s \in \bigcap f[P(T) \setminus \rho_T(t)] \}\right). \]

Define a map
\[ \Phi' : \text{Slat}(S, T) \to (S^\sigma_\Lambda)^{P(T)} \]
as follows: for \( g \in \text{Slat}(S, T) \) and \( F \in P(T) \), let
\[ [\Phi'(g)](F) := g^{-1}(T \setminus F). \]

Then \( \Psi' \) and \( \Phi' \) are mutually-inverse order-isomorphisms. The restriction of \( \Psi' \) to \((S^\sigma)^{P(T)}\) maps onto \( \text{Slat}^{\text{fin}}(S, T) \).

It has been shown that if \( L \) is a finite lattice, then \( \text{Slat}(L, L) \in \mathbf{D} \) if and only if \( L \in \mathbf{D} \) (see [14], Theorem 3). Indeed, if \( L \in \mathbf{D} \) and \( L \) is finite, [14], Lemma 1 states that \( L^{\mathcal{J}(L)} \cong \text{Slat}(L, L) \). We also have the following
Corollary 3.9. Let $L \in \mathcal{D}$ be finite. Then
\[
L^\mathcal{J}(L) \cong \left( \text{Slat}(L, L) \right)^\vartheta.
\]
Therefore Slat($L, L$) is self-dual.

Proof. As $L$ is finite, $L^\vartheta \cong L$ and $P(L) = \{ \uparrow j \mid j \in \mathcal{J}(L) \} \cong \mathcal{J}(L)^\vartheta$. 

Indeed, Slat($L, L$) is the coproduct of $L$ and $L^\vartheta$ in $\mathcal{D}$ for a finite distributive lattice $L$. (See [5], Corollary 2.3; [6], Theorem and Corollary; and [24], Theorem.)

Under Priestley duality, continuous order-preserving maps between Priestley spaces $P$ and $Q$ correspond to $\{0, 1\}$-preserving homomorphisms between $D(Q)$ and $D(P)$. In [3], $\{0\}$-$\vee$-homomorphisms were shown to correspond to certain relations between $P$ and $Q$.

Let $P, Q \in \mathcal{P}$; let $R \subseteq P \times Q$. For $p \in P$, $R(p) := \{ q \in Q \mid (p, q) \in R \}$. For $V \subseteq Q$, $R^{-1}(V) := \{ p \in P \mid R(p) \cap V \neq \emptyset \}$. The relation $R$ is a Priestley relation if
1. $R(p)$ is a closed down-set of $Q$ for all $p \in P$;
2. $R^{-1}(V) \in D(P)$ for all $V \in D(Q)$.

Let $\mathcal{R}(P, Q)$ denote the set of Priestley relations from $P$ to $Q$.

For $R \in \mathcal{R}(P, Q)$, let $R^*: D(Q) \rightarrow D(P)$ be the function $R^*(V) = R^{-1}(V)$ ($V \in D(Q)$). By [3], Lemma 1.5, it is a $\{0\}$-$\vee$-homomorphism. Indeed, the map
\[
R \mapsto R^* \quad [R \in \mathcal{R}(P, Q)]
\]

is a bijection between $\mathcal{R}(P, Q)$ and Slat($D(Q), D(P)$) (where we regard $D(P)$ and $D(Q)$ as $\{\emptyset\}$-$\cup$-semilattices).

We shall turn $\mathcal{R}(P, Q)$ into a poset as follows: for $R, S \in \mathcal{R}(P, Q)$, $R \leq S$ if and only if $R(p) \subseteq S(p)$ for all $p \in P$.

Lemma 3.10. Let $P, Q \in \mathcal{P}$, $R \in \mathcal{R}(P, Q)$. Then for all $p \in P$,
\[
Q \setminus R(p) = \bigcup \{ V \in D(Q) \mid p \notin R^*(V) \}.
\]

Proof. For all $p \in P$, the set $Q \setminus R(p) \in \mathcal{U}(Q)$. By the isomorphism of §2,
\[
Q \setminus R(p) = \bigcup \{ V \in D(Q) \mid V \subseteq Q \setminus R(p) \}
= \bigcup \{ V \in D(Q) \mid R(p) \subseteq Q \setminus V \}
= \bigcup \{ V \in D(Q) \mid p \notin R^{-1}(V) \}.
\]
Lemma 3.11. Let \( P, Q \in \mathbf{P} \). The map
\[
R \mapsto R^* \quad [R \in \mathcal{R}(P, Q)]
\]
from \( \mathcal{R}(P, Q) \) to \( \text{Slat} \left[ \left( D(Q), \cup, \emptyset \right), \left( D(P), \cup, \emptyset \right) \right] \) is an order-isomorphism.

**Proof.** Let \( R, S \in \mathcal{R}(P, Q) \). First assume \( R \subseteq S \). Then for all \( V \in D(Q) \)
\[
R^*(V) = R^{-1}(V) = \{ p \in P \mid R(p) \cap V \neq \emptyset \} \\
\subseteq \{ p \in P \mid S(p) \cap V \neq \emptyset \} = S^{-1}(V) = S^*(V).
\]
Therefore \( R^* \leq S^* \). Hence the map is order-preserving.

Now assume \( R^* \leq S^* \). By Lemma 10, for all \( p \in P \),
\[
Q \setminus S(p) = \bigcup \{ V \in D(Q) \mid p \notin S^*(V) \} \\
\subseteq \bigcup \{ V \in D(Q) \mid p \notin R^*(V) \} = Q \setminus R(p),
\]
so that \( R(p) \subseteq S(p) \). Therefore \( R \subseteq S \). Hence the map is an order-embedding.

As the map is onto, it is an order-isomorphism.

Now we establish the connection between our function space representation of \( \text{Slat} \)-morphisms and Priestley relations.

**Proposition 3.12.** Let \( P, Q \in \mathbf{P} \). We regard \( D(P) \) and \( D(Q) \) as \( \{ \emptyset \} \)-\( \cup \)-semilattices. Define
\[
\Theta: \mathcal{R}(P, Q) \to \mathcal{U}(Q)^{P^0}_\Lambda
\]
as follows: for \( R \in \mathcal{R}(P, Q) \) and \( p \in P \), let
\[
[\Theta(R)](p) := Q \setminus R(p).
\]
Define
\[
\Psi': \mathcal{U}(Q)^{P^0}_\Lambda \to \text{Slat} \left( D(Q), D(P) \right)
\]
as follows:
\[
[\Psi'(f)](V) := P \setminus f^{-1}(I_{\mathcal{U}(Q)} V) \quad [f \in \mathcal{U}(Q)^{P^0}_\Lambda, V \in D(Q)].
\]
Define
\[
\Phi': \text{Slat} \left( D(Q), D(P) \right) \to \mathcal{U}(Q)^{P^0}_\Lambda
\]
as follows: for $g \in \text{Slat}(D(Q), D(P))$ and $p \in P$, let

$$[\Phi'(g)](p) := \bigcup \{ V \in D(Q) \mid p \notin g(V) \}.$$ 

Then:

1. $\theta$ is a dual-isomorphism;
2. $\Psi'$ and $\Phi'$ are mutually-inverse dual-isomorphisms;
3. For all $R \in \mathcal{R}(P, Q)$,
   $$(\Psi' \circ \theta)(R) = R^*.$$ 

Proof. By Theorem 6, $\Psi'$ and $\Phi'$ are inverse dual-isomorphisms.

For $R \in \mathcal{R}(P, Q)$ and $p \in P$,

$$[\Phi'(R^*)](p) = \bigcup \{ V \in D(Q) \mid p \notin R^*(V) \} = Q \setminus R(p)$$

by Lemma 10. Hence $\theta$ is well-defined and $\Phi'(R^*) = \theta(R)$, so

$$(\Psi' \circ \theta)(R) = R^*.$$ 

4. Profinite posets and Priestley powers

In [28], Theorem, it is shown that every Priestley space $P$ is an inverse limit of finite posets with the discrete topology. Although the proof requires minor modifications, the basic idea is to partition the space into finitely many parts and place a partial order on the set of equivalence classes (if possible) so that the natural projection map is continuous and order-preserving. The inverse limit of the filtered system of these ordered partitions will be the original Priestley space (Proposition 6). If one does this same procedure with $P$, a priori one will get more partitions. We show, however, that $P$ is in fact the inverse limit of the unordered versions of the partitions arising from $P$ (Proposition 7).

If $M \in \text{Lat} \cup \text{Slat}$, then, for each of the above partitions $\Pi$ of $P$, one gets a Priestley power $M^\Pi$, and the filtered limit of these is $M^P$ (Proposition 14). For $P$, however, we are not using all the partitions that arise from $P$ necessarily, but only those arising from $P$. While the inverse limit of each filtered system of partitions (the one arising from $P$, the other from $P$) is $P$, we must prove that the corresponding filtered limit is $M^P$ (Proposition 15). We use a lemma, interesting in its own right, to show that if a Priestley space is an inverse limit of two filtered systems of finite antichains, then any partition arising from one system may be refined to yield a partition arising from the other system (Lemma 9).

The first lemmas are easy.
Lemma 4.1. Let $P$ be a set, $\Pi, \Pi_0$ partitions of $P$ such that $\Pi \leq \Pi_0$. Then

$$\nu_{\Pi_0} = \nu_{\Pi_0} \circ \nu_{\Pi}.$$  

Lemma 4.2. Let $P \in \mathcal{P}$. Then:

1. every $\Pi \in \mathcal{E}_P$ is a finite poset, the elements of which are non-empty clopen subsets of $P$;
2. for all $\Pi \in \mathcal{E}_P$, $\nu_{\Pi}: P \to \Pi$ is continuous, order-preserving, and surjective;
3. for all $\Pi, \Pi_0 \in \mathcal{E}_P$ such that $\Pi \leq \Pi_0$,

$$\nu_{\Pi_0} \circ \nu_{\Pi}: \Pi \to \Pi_0$$

is order-preserving and surjective;
4. $\mathcal{E}_P \subseteq \mathcal{E}_P$.

Lemma 4.3. Let $P \in \mathcal{P}$; let $Q$ be a poset. Let $f \in Q^P$. For each $q \in \text{Im } f$, let $V_q := f^{-1}(q)$; let $\Pi := \{V_q\}_{q \in \text{Im } f}$. Define $g: \Pi \to Q$ by $g(V_q) := q$ for all $q \in \text{Im } f$. Then $\Pi \in \mathcal{E}_P$, $g$ is order-preserving, and $f = g \circ \nu_{\Pi}$.

**Proof.** Clearly $\Pi$ is a partition of $P$ into open subsets. We now prove that the quasiorder $\leq_{\Pi}$ is antisymmetric. Let $q, r \in \text{Im } f$. Assume that $V_q \leq_{\Pi} V_r$. Then for some $n \geq 1$, there exist $q_1, \ldots, q_n \in \text{Im } f$ such that

$$V_q = V_{q_1} \leq_{\Pi} \cdots \leq_{\Pi} V_{q_n} = V_r.$$  

As $f$ is order-preserving,

$$q = q_1 \leq \cdots \leq q_n = r,$$

so $q \leq r$. Thus, if $q, r \in \text{Im } f$, $V_q \leq_{\Pi} V_r$, and $V_r \leq_{\Pi} V_q$, then $q = r$ and hence $V_q = V_r$. Therefore $\leq_{\Pi}$ is antisymmetric. We conclude that $\Pi \in \mathcal{E}_P$.

The above shows that $g$ is order-preserving and clearly $f = g \circ \nu_{\Pi}$. \hfill $\blacksquare$

Proposition 4.4. Let $P \in \mathcal{P}$. The poset $\mathcal{E}_P$ is filtered.
Proof. If $P = \emptyset$, the partition with no equivalence classes is in $\mathcal{E}_P$. If $P \neq \emptyset$, the partition $\{P\} \in \mathcal{E}_P$. In either case, $\mathcal{E}_P \neq \emptyset$.

Now let $\Pi_1, \Pi_2 \in \mathcal{E}_P$. By Lemma 2 (2),

$$
\nu_i := \nu_{\Pi_i} : P \to \Pi_i \quad (i = 1, 2)
$$

is continuous and order-preserving. Thus the map $\nu : P \to \Pi_1 \times \Pi_2$ defined by $\nu(p) := \left(\nu_1(p), \nu_2(p)\right) \ (p \in P)$ is a continuous order-preserving map into an ordered space with the discrete topology. For $q \in \text{Im} \nu$, let $V_q := \nu^{-1}(q)$. By Lemma 3,

$$
\Pi := \{V_q\}_{q \in \text{Im} \nu} \in \mathcal{E}_P.
$$

Clearly $\Pi \leq \Pi_1, \Pi_2$.

Lemma 4.5. Let $P \in \mathcal{P}$.

1. If $U \in \mathcal{D}(P)$ is non-empty and proper, then $\{U, P \setminus U\} \in \mathcal{E}_P$.

2. If $p, q \in P$ and $p \nleq q$, then there exists $\Pi \in \mathcal{E}_P$ such that $\nu_\Pi(p) \nleq \nu_\Pi(q)$.

Proof. (1) This part is obvious.

(2) There exists $U \in \mathcal{D}(P)$ such that $p \in U$ and $q \in P \setminus U$. Let $\Pi := \{U, P \setminus U\}$.

Proposition 4.6 ([28], Theorem). Let $P \in \mathcal{P}$. Then

$$
\left(\{\Pi\}_{\Pi \in \mathcal{E}_P}, (\nu_{\Pi_1, \Pi_2} : \Pi_1 \to \Pi_2)^{n_1, n_2 \in \mathcal{E}_P}_{n_1 \leq n_2}\right)
$$

is an inverse system in $\mathcal{P}$ with inverse limit

$$
\left(P, (\nu_\Pi : P \to \Pi)^{\Pi \in \mathcal{E}_P}\right).
$$

Proof. By Proposition 4, $\mathcal{E}_P$ is filtered, and it is clear from Lemma 2 that

$$
\mathcal{T} := \left(\{\Pi\}_{\Pi \in \mathcal{E}_P}, (\nu_{\Pi_1, \Pi_2} : \Pi_1 \to \Pi_2)^{n_1, n_2 \in \mathcal{E}_P}_{n_1 \leq n_2}\right)
$$

is an inverse system in $\mathcal{P}$. By Lemma 1,

$$
\left(P, (\nu_\Pi : P \to \Pi)^{\Pi \in \mathcal{E}_P}\right).
$$

is compatible with $\mathcal{T}$. 

---


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Assume
\[ (Q, (g_{\Pi})_{\Pi \in E}) \]
is also compatible with \( T \). We prove that, for each \( q \in Q \),
\[ \bigcap_{\Pi \in E} \nu_{\Pi}^{-1}(g_{\Pi}(q)) \]
is a singleton.

By Lemma 2 (2), \( C_\Pi := \nu_{\Pi}^{-1}(g_{\Pi}(q)) \) is clopen and non-empty for all \( \Pi \in E \).
If \( \Pi_1, \ldots, \Pi_n \in E \) for some \( n \geq 0 \), there exists \( \Pi \in E \) such that \( \Pi \leq \Pi_1, \ldots, \Pi_n \).
As \( C_\Pi \neq \emptyset \), there exists \( p \in P \) such that \( \nu_{\Pi}(p) = g_{\Pi}(q) \).
By Lemma 1, for \( i = 1, \ldots, n \),
\[ \nu_{\Pi_i}(p) = (\nu_{\Pi_i} \circ \nu_{\Pi})(p) = (\nu_{\Pi_i} \circ g_{\Pi})(q) = g_{\Pi_i}(q) \]
by compatibility, so
\[ p \in \bigcap_{i=1}^{n} C_{\Pi_i}. \]

By compactness,
\[ \bigcap_{\Pi \in E} C_\Pi \neq \emptyset. \]

If \( p, p' \in P \) and \( p \neq p' \), by Lemma 5 (2) there exists \( \Pi \in E \) such that \( \nu_{\Pi}(p) \neq \nu_{\Pi}(p') \).
Hence
\[ \bigcap_{\Pi \in E} C_\Pi \]
contains a unique element \( g(q) \).

We prove that \( g : Q \to P \) is continuous and order-preserving. Let \( U \in D(P) \) be non-empty and proper. Let \( p \in U \). Then \( \Pi := \{ U, P \setminus U \} \in E \) by Lemma 5 (1) and \( g_{\Pi}^{-1}(\nu_{\Pi}(p)) = g_{\Pi}^{-1}({U}) \in D(Q) \).
We have
\[ \nu_{\Pi} \circ g = g_{\Pi} \implies g^{-1} \circ \nu_{\Pi}^{-1} = g_{\Pi}^{-1} \]
\[ \implies g^{-1} \circ \nu_{\Pi}^{-1} \circ \nu_{\Pi} = g_{\Pi}^{-1} \circ \nu_{\Pi} \]
\[ \implies g^{-1}(U) = (g^{-1} \circ \nu_{\Pi}^{-1} \circ \nu_{\Pi})(p) = (g_{\Pi}^{-1} \circ \nu_{\Pi})(p) \in D(Q). \]
Hence \( g : Q \to P \) is order-preserving and continuous. Uniqueness is clear. \( \blacksquare \)
Proposition 4.7. Let \( P \in \mathcal{P} \). Then
\[
\left( (\Pi)_{\Pi \in \mathcal{E}_P}, \left( \nu_{\Pi_1, \Pi_2}: \Pi_1 \to \Pi_2 \right)_{n_1, n_2 \in \mathcal{E}_P} \right)
\]
is an inverse system in \( \mathcal{P} \) with inverse limit
\[
\left( \mathcal{T}, \left( \nu_{\mathcal{T}, \Pi} \to \Pi \right)_{\Pi \in \mathcal{E}_P} \right).
\]

Proof. Using Proposition 6, we see that
\[
\mathcal{T} := \left( (\Pi)_{\Pi \in \mathcal{E}_P}, \left( \nu_{\Pi_1, \Pi_2}: \Pi_1 \to \Pi_2 \right)_{n_1, n_2 \in \mathcal{E}_P} \right)
\]
is an inverse system in \( \mathcal{P} \) with which \( \left( \mathcal{T}, \left( \nu_{\mathcal{T}, \Pi} \to \Pi \right)_{\Pi \in \mathcal{E}_P} \right) \) is compatible.

Assume \( (Q, (\bar{g}_{\Pi}: Q \to \Pi)_{\Pi \in \mathcal{E}_P}) \) is also compatible with \( \mathcal{T} \). For each \( \Pi \in \mathcal{E}_P \), let \( g_{\Pi}: Q \to \Pi \) be the continuous order-preserving function \( g_{\Pi}(q) := \bar{g}_{\Pi}(q) \) (\( q \in Q \)). Then
\[
\left( Q, (g_{\Pi}: Q \to \Pi)_{\Pi \in \mathcal{E}_P} \right)
\]
is compatible with the inverse system
\[
\left( (\Pi)_{\Pi \in \mathcal{E}_P}, \left( \nu_{\Pi_1, \Pi_2}: \Pi_1 \to \Pi_2 \right)_{n_1, n_2 \in \mathcal{E}_P} \right)
\]
(see Proposition 6). Hence there is a unique continuous order-preserving function \( g: Q \to P \) such that \( \nu_{\Pi} \circ g = g_{\Pi} \) for all \( \Pi \in \mathcal{E}_P \).

For all \( q, r \in Q \), \( q \leq r \) implies \( g(q) = g(r) \). For otherwise by Lemma 5 there exists \( \Pi \in \mathcal{E}_P \) such that \( \nu_{\Pi}(g(q)) \not\geq \nu_{\Pi}(g(r)) \) so that \( g_{\Pi}(q) \not\geq g_{\Pi}(r) \) and hence \( g_{\Pi}(q) \not\geq g_{\Pi}(r) \). As \( \Pi \) is an antichain, we have \( g_{\Pi}(q) \not\geq g_{\Pi}(r) \), so that \( g_{\Pi} \) is not order-preserving, a contradiction.

Hence the map \( \bar{g}: Q \to \mathcal{T} \) defined by \( \bar{g}(q) := g(q) \) for all \( q \in Q \) is continuous and order-preserving. Moreover for all \( \Pi \in \mathcal{E}_P \), \( \nu_{\mathcal{T}, \Pi} \circ \bar{g} = \bar{g}_{\Pi} \).

Assume \( \bar{h}: Q \to \mathcal{T} \) is a continuous order-preserving map such that \( \nu_{\mathcal{T}, \Pi} \circ \bar{h} = \bar{g}_{\Pi} \) (\( \Pi \in \mathcal{E}_P \)). Define \( h: Q \to P \) by \( h(q) := \bar{h}(q) \) (\( q \in Q \)). Then \( h \) is continuous and order-preserving, and \( \nu_{\Pi} \circ h = g_{\Pi} \) (\( \Pi \in \mathcal{E}_P \)); hence \( h = g \), so that \( \bar{h} = \bar{g} \).
Lemma 4.8. Let $F$ be a filtered poset, \( \left( P, (g_i: P \to P_i)_{i \in F} \right) \) an inverse limit in $P$ of the inverse system 
\[ \left( (P_i)_{i \in F}, (g_{ij}: P_i \to P_j)_{i \leq j \in F} \right). \]

Then:
1. $i \leq j$ implies $\text{Im } D(g_j) \subseteq \text{Im } D(g_i)$ (\( i, j \in F \));
2. $\{ \text{Im } D(g_i) \mid i \in F \}$ is directed;
3. $D(P) = \bigcup_{i \in F} \text{Im } D(g_i)$.

Proof. (1) Let $i, j \in F$ be such that $i \leq j$. Then $g_{ij} \circ g_i = g_j$ implies

\[ D(g_i) \circ D(g_{ij}) = D(g_j), \]

so that $\text{Im } D(g_j) \subseteq \text{Im } D(g_i)$.

(2) This statement follows from (1) and the fact $F$ is filtered.

(3) By Priestley duality,

\[ \mathcal{S} := \left( \left( D(P_i) \right)_{i \in F}, \left( D(g_{ij}): D(P_j) \to D(P_i) \right)_{i \leq j \in F} \right) \]

is a filtered system in $D$ with filtered limit

\[ \left( D(P), \left( D(g_i): D(P_i) \to D(P) \right)_{i \in F} \right). \]

Let $\mathcal{D} := \{ \text{Im } D(g_i) \mid i \in F \}$. Then $M := \bigcup \mathcal{D}$ is a $\{0,1\}$-sublattice of $L := D(P)$ by (2). For $i \in F$, let $f'_i: D(P_i) \to M$ be the $\{0,1\}$-homomorphism defined by $f'_i(a) := [D(g_i)](a)$ \( (a \in D(P_i)) \). For $i, j \in F$ such that $i \leq j$ and $a \in D(P_j)$,

\[ [f'_{ij} \circ D(g_{ij})](a) = [D(g_i) \circ D(g_{ij})](a) = [D(g_{ij} \circ g_i)](a) \]

so that \( \left( M, (f'_i: D(P_i) \to M)_{i \in F} \right) \) is compatible with $\mathcal{S}$. Hence there exists a unique $\{0,1\}$-homomorphism $f: L \to M$ such that $f \circ D(g_i) = f'_i$ (\( i \in F \)). For all $i \in F$ and $a \in D(P_i)$, $[f \circ D(g_i)](a) = f'_i(a) = [D(g_i)](a)$.

Let $h: L \to L$ be the $\{0,1\}$-homomorphism defined by $h(a) := f(a)$ (\( a \in L \)). As $h \circ D(g_i) = D(g_i)$ (\( i \in F \)), we see that $h = \text{id}_L$, so that $\text{Im } f = L$ and hence $M = L$. 

\[ \square \]
Lemma 4.9. Let $F$ and $K$ be filtered posets. Let $\left( P, (g_i: P_i \to P)_{i \in F} \right)$ be an inverse limit in $P$ of the inverse system $\left( P_i \right)_{i \in F}$.

Let $\left( Q_k \right)_{k \in K}$ be an inverse limit in $P$ of the inverse system $\left( Q_k \right)_{k \in K}$.

Assume that $g_i: P_i \to P$ and $h_k: Q_k \to Q$ are surjective and that $P_i$ and $Q_k$ are finite antichains ($i \in F$, $k \in K$).

Then for all $k \in K$, there exists $i \in F$ for which the following holds: for all $p_i \in P_i$, there exists $q_k \in Q_k$ such that $g_i^{-1}(p_i) \subseteq h_k^{-1}(q_k)$.

Proof. Let $k \in K$. By Lemma 8 (3), $\text{Im } D(h_k) \subseteq \bigcup_{i \in F} \text{Im } D(g_i)$. Hence by Lemma 8 (2) there exists $i \in F$ such that $\text{Im } D(h_k) \subseteq \text{Im } D(g_i)$.

As $\text{Im } D(h_k)$ is a $\{0, 1\}$-sublattice of $\text{Im } D(g_i)$,

$$a \leq 1_{D(P)} = \bigvee \{ b \in D(P) \mid b \text{ is an atom of } \text{Im } D(h_k) \}$$

for every atom $a$ of $\text{Im } D(g_i)$, so there exists an atom $b \in \text{Im } D(h_k)$ such that $a \leq b$. That is, for every $p_i \in P_i$, there exists $q_k \in Q_k$ such that $g_i^{-1}(p_i) \subseteq h_k^{-1}(q_k)$.

The next result is easily seen to be true.

Lemma 4.10. Let $P, Q \in P$, $M \in \text{Lat} \cup \text{Slat}$. Let $\nu: P \to Q$ be a continuous order-preserving map. Define $\mu: M^Q \to M^P$ by $\mu(f) := f \circ \nu$ for all $f \in M^Q$.

Then:

1. $\mu$ is a morphism;
2. $\mu$ is injective if $\nu$ is surjective.

Lemma 2 (2) and (3) and Lemma 10 yield the following.

Lemma 4.11. Let $P \in P$, $M \in \text{Lat} \cup \text{Slat}$.

1. For every $\Pi \in \mathcal{E}_P$, $\mu^M_\Pi: M^{\Pi} \to M^P$ is an injective morphism;
2. For every $\Pi, \Pi_0 \in \mathcal{E}_P$ such that $\Pi \leq \Pi_0$,

$$\mu^M_{\Pi, \Pi_0}: M^{\Pi_0} \to M^{\Pi}$$

is an injective morphism.
Lemma 4.12. Let $P \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$. 

(1) For $\Pi, \Pi' \in \mathcal{E}_{P}$, $\mu_{\Pi,\Pi'}^{M} = \text{id}(M)$. 

(2) For $\Pi_{1}, \Pi_{2}, \Pi_{3} \in \mathcal{E}_{P}$ such that $\Pi_{1} \leq \Pi_{2} \leq \Pi_{3}$,

\[ \mu_{\Pi_{1},\Pi_{2}}^{M} \circ \mu_{\Pi_{2},\Pi_{3}}^{M} = \mu_{\Pi_{1},\Pi_{3}}^{M}. \]

(3) For $\Pi_{1}, \Pi_{2} \in \mathcal{E}_{P}$ such that $\Pi_{1} \leq \Pi_{2}$,

\[ \mu_{\Pi_{1}}^{M} \circ \mu_{\Pi_{1},\Pi_{2}}^{M} = \mu_{\Pi_{2}}^{M}. \]

Proof. (1) This part is obvious.

(2) Let $f \in M^{\Pi_{3}}$. Then

\[
(\mu_{\Pi_{1},\Pi_{2}}^{M} \circ \mu_{\Pi_{2},\Pi_{3}}^{M})(f) = f \circ \nu_{\Pi_{2},\Pi_{3}} \circ \nu_{\Pi_{1},\Pi_{2}} = f \circ \nu_{\Pi_{1},\Pi_{3}} = \mu_{\Pi_{1},\Pi_{3}}^{M}(f).
\]

(3) Let $f \in M^{\Pi_{3}}$. Then

\[
(\mu_{\Pi_{1}}^{M} \circ \mu_{\Pi_{1},\Pi_{2}}^{M})(f) = f \circ \nu_{\Pi_{1},\Pi_{2}} \circ \nu_{\Pi_{1}} = f \circ \nu_{\Pi_{2}} = \mu_{\Pi_{2}}^{M}(f)
\]

by Lemma 1.

Lemma 4.13. Let $F$ be a filtered poset. Let

\[ S := \left((C_{i})_{i \in F}, (f_{ij}: C_{j} \rightarrow C_{i})_{i,j \in F}\right) \]

be a filtered system in $\mathbf{Lat} \cup \mathbf{Slat}$ with which $\left(C, (f_{i}: C_{i} \rightarrow C)_{i \in F}\right)$ is compatible. Assume:

(1) $C = \bigcup_{i \in F} \text{Im} f_{i}$;

(2) for all $i \in F$, $f_{i}$ is injective.

Then $\left(C, (f_{i}: C_{i} \rightarrow C)_{i \in F}\right)$ is a filtered limit of $S$. 

---


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Proof. Assume \( (C', (f'_i; C_i \to C')_{i \in F}) \) is compatible with \( S \). Define

\[ f: C \to C' \]

as follows: if \( c \in C \) and \( c = f_i(c_i) \) for some \( i \in F \) and \( c_i \in C_i \), let \( f(c) := f'_i(c_i) \in C' \).

The map is well-defined. For if \( c = f_j(c_j) = f_k(c_k) \) for some \( j, k \in F \), \( c_j \in C_j \), \( c_k \in C_k \), there exists \( i \in F \) such that \( i \leq j, k \). Hence

\[ c = (f_i \circ f_{ij})(c_j) = (f_i \circ f_{ik})(c_k), \]

so that \( f_{ij}(c_j) = f_{ik}(c_k) \). Now \( (f'_i \circ f_{ij})(c_j) = f'_j(c_j) \) and \( (f'_i \circ f_{ik})(c_k) = f'_k(c_k) \).

If \( c, d \in C \), then there exist \( j, k \in F \) such that \( c = f_j(c_j) \) and \( d = f_k(c_k) \) for some \( c_j \in C_j \) and \( c_k \in C_k \). There exists \( i \in F \) such that \( i \leq j, k \), and \( c = (f_i \circ f_{ij})(c_j) \), \( d = (f_i \circ f_{ik})(c_k) \). Thus \( c \vee d = f_i \left( f_{ij}(c_j) \vee f_{ik}(c_k) \right) \), so

\[ f(c \vee d) = f'_i \left( f_{ij}(c_j) \vee f_{ik}(c_k) \right) \]

\[ = (f'_i \circ f_{ij})(c_j) \vee (f'_i \circ f_{ik})(c_k) \]

\[ = f'_j(c_j) \vee f'_k(c_k) = f(c) \vee f(d). \]

(If \( f \in \text{Lat} \), then it preserves meet as well.) Hence \( f \) is a morphism. Uniqueness is clear.

\[ \text{Cf. (2) below with [18], Theorem V.4.1.} \]

Proposition 4.14. Let \( P \in \mathcal{P} \), \( M \in \text{Lat} \cup \text{Slat} \). Then:

1. \( M^P = \bigcup_{\Pi \in \mathcal{E}_P} \text{Im} \mu^M_{\Pi} ; \)

2. \( \left( M^P, (\mu^M_{\Pi}: M^{\Pi} \to M^P)_{\Pi \in \mathcal{E}_P} \right) \) is a filtered limit of the filtered system

\( \left( (M^{\Pi})_{\Pi \in \mathcal{E}_P}, (\mu^M_{\Pi_1, \Pi_2}: M^{\Pi_2} \to M^{\Pi_1})_{\Pi_1, \Pi_2 \in \mathcal{E}_P} \right) ; \)

3. for \( \Pi_1, \Pi_2 \in \mathcal{E}_P, \Pi_1 \leq \Pi_2 \) implies

\[ \text{Im} \mu^M_{\Pi_2} \subseteq \text{Im} \mu^M_{\Pi_1} . \]
Proof. (1) Let \( f \in M^P \). By Lemma 3, there exist \( \Pi \in \mathcal{E}_P \) and an order-preserving map \( g: \Pi \to M \) such that \( f = g \circ \nu_{\Pi} = \mu_{\Pi}^M(g) \).

(2) By Proposition 4, \( \mathcal{E}_P \) is filtered. By Lemma 12,
\[
\mathcal{S} := \left( (M^\Pi)_{\Pi \in \mathcal{E}_P}, (\mu_{\Pi_1, \Pi_2}^M: M^{\Pi_2} \to M^{\Pi_1})_{n_1, n_2 \in \mathcal{E}_P} \right)
\]
is a filtered system with which
\[
\left( M^P, (\mu_{\Pi}^M: M^\Pi \to M^P)_{\Pi \in \mathcal{E}_P} \right)
\]
is compatible. By (1) and Lemmas 11 and 13, it is a filtered limit of \( \mathcal{S} \).

(3) This part follows from (2).

Proposition 4.15. Let \( P \in \mathcal{P} \), \( M \in \text{Lat} \cup \text{Slat} \). Then:

1. \( M^P = \bigcup_{\Pi \in \mathcal{E}_P} \operatorname{Im} \mu_{\Pi}^M \);

2. \( \left( M^\Pi, (\mu_{\Pi}^M: M^{\Pi} \to M^P)_{\Pi \in \mathcal{E}_P} \right) \) is a filtered limit of the filtered system
\[
\left( (M^\Pi)_{\Pi \in \mathcal{E}_P}, (\mu_{\Pi_1, \Pi_2}^M: M^{\Pi_2} \to M^{\Pi_1})_{n_1, n_2 \in \mathcal{E}_P} \right);
\]

3. for \( \Pi_1, \Pi_2 \in \mathcal{E}_P, \Pi_1 \preceq \Pi_2 \) implies
\[
\operatorname{Im} \mu_{\Pi_2}^M \subseteq \operatorname{Im} \mu_{\Pi_1}^M.
\]

Proof. (1) Let \( f \in M^P \). By Lemma 3, there exist \( \Pi_0 \in \mathcal{E}_P \) and a map \( g \in M^{\Pi_0} \) such that
\[
f = g \circ \nu_{\Pi_0}.
\]

By Propositions 6 and 7 and Lemma 9, there exists \( \Pi \in \mathcal{E}_P \) such that \( \Pi \preceq \Pi_0 \). By Lemma 1,
\[
f = g \circ \nu_{\Pi_0} = g \circ \nu_{\Pi, \Pi_0} \circ \nu_{\Pi} \in \operatorname{Im} \mu_{\Pi}^M.
\]

(2) By Proposition 14,
\[
\mathcal{S} := \left( (M^\Pi)_{\Pi \in \mathcal{E}_P}, (\mu_{\Pi_1, \Pi_2}^M: M^{\Pi_2} \to M^{\Pi_1})_{n_1, n_2 \in \mathcal{E}_P} \right)
\]
is a filtered system with which
\[
\left( M^P, (\mu_{\Pi}^M: M^\Pi \to M^P)_{\Pi \in \mathcal{E}_P} \right)
\]
is compatible.

By (1) and Lemmas 11 and 13, it is a filtered limit of \( S \).

(3) This part follows from (2).

### 5. Compact congruences of Priestley powers

In this section we prove that, for \( L \in \text{Lat} \) and \( P \in \text{P} \), \( \text{Comp}(L^P) \cong (\text{Comp} L)^P \) (Theorem 10). We use a suggestion from [23], pp. 98–100 about obtaining the semilattice of compact congruences of a limit of lattices \( L_i \) as a limit of the semilattices \( \text{Comp} L_i \).

In [23], p. 98 and [11], §3, two prescriptions are given for functors from \( \text{Lat} \) to \( \text{Slat} \) given by \( L \mapsto \text{Comp} L \) (\( L \in \text{Lat} \)). We show that these two prescriptions yield the same functor (Lemma 3).

Our first lemma is a consequence of [19], Theorem 1.20. The second is a corollary, but we use an easy proof suggested by Dr. P. M. Neumann.

**Lemma 5.1.** Let \( L \in \text{Lat} \), \( X \subseteq L \times L \). Define \( A_n(X) \) \((n \geq 0)\) by induction:

\[
A_0(X) := X \cup \{(a, b) \mid (b, a) \in X\} \cup \{(a, a) \mid a \in L\};
\]

\[
A_{n+1}(X) := A_n(X) \cup Q_n(X) \cup T_n(X)
\]

where

\[
Q_n(X) := \{(a_1 \lor a_2, b_1 \lor b_2), (a_1 \land a_2, b_1 \land b_2) \mid (a_i, b_i) \in A_n(X) \ (i = 1, 2)\},
\]

\[
T_n(X) := \{(a, c) \mid (a, b), (b, c) \in A_n(X) \text{ for some } b \in L\}.
\]

Then \( \vartheta^L(X) = \bigcup_{n \geq 0} A_n(X) \).

**Lemma 5.2.** Let \( L, M \in \text{Lat} \), \( X \subseteq L \times L \). Let \( f : L \rightarrow M \) be a homomorphism. Then

\[
(f \times f)[\vartheta^L(X)] \subseteq \vartheta^M\left((f \times f)[X]\right).
\]

**Proof.** Let \( \rho := \vartheta^M\left((f \times f)[X]\right) \), and let \( \varsigma := (f \times f)^{-1}(\rho) \). Since \( f \) is a homomorphism, \( \varsigma \in \text{Con} L \). Clearly \( X \subseteq \varsigma \), so \( \vartheta^L(X) \subseteq \varsigma \). Hence

\[
(f \times f)[\vartheta^L(X)] \subseteq \vartheta^M\left((f \times f)[X]\right).
\]

\( \blacksquare \)
Lemma 5.3. Let \( L, M \in \text{Lat} \) and let \( f: L \to M \) be a homomorphism. Let \( n \geq 0 \), \( a_i, b_i \in L \ (i = 1, \ldots, n) \). Then
\[
[\text{Comp}(f)] \left( \bigvee_{i=1}^{n} \vartheta^L(a_i, b_i) \right) = \bigvee_{i=1}^{n} \vartheta^M(f(a_i), f(b_i)).
\]
Hence \( \text{Comp} \) is a functor from \( \text{Lat} \) to \( \text{Slat} \).

Proof. Clearly \( \bigvee_{i=1}^{n} \vartheta^M(f(a_i), f(b_i)) \subseteq [\text{Comp}(f)] \left( \bigvee_{i=1}^{n} \vartheta^L(a_i, b_i) \right) \). By Lemma 2,
\[
(f \times f)[\bigvee_{i=1}^{n} \vartheta^L(a_i, b_i)] \subseteq \bigvee_{i=1}^{n} \vartheta^M(f(a_i), f(b_i)),
\]
so that
\[
[\text{Comp}(f)] \left( \bigvee_{i=1}^{n} \vartheta^L(a_i, b_i) \right) \subseteq \bigvee_{i=1}^{n} \vartheta^M(f(a_i), f(b_i)).
\]
Thus
\[
[\text{Comp}(f)] \left( \bigvee_{i=1}^{n} \vartheta^L(a_i, b_i) \right) = \bigvee_{i=1}^{n} \vartheta^M(f(a_i), f(b_i)).
\]

In [9], Theorem 2.1, it is proven that, for \( L \in \text{Lat}, P \in \text{Pfin} \), \( \text{Con}(L^P) \cong (\text{Con} L)^n \), where \( n \) is the cardinality of \( P \). (Also see a similar result for certain lattice-ordered algebras, [8], Theorem 3.5.) The proof is by induction on \( n \). We present essentially the same proof below, only we have made it direct.

First we state some lemmas.

Lemma 5.4. Let \( L \in \text{Lat}, P \in \text{Pfin}, a, b \in L, p \in P \). Then
\[
m_P(a, b, p) \in L^P
\]

Lemma 5.5. Let \( A, B \) be algebraic lattices. Then \( \kappa(A \times B) = \kappa(A) \times \kappa(B) \). For \( n \geq 0 \), \( \kappa(A^n) = \kappa(A)^n \).

Lemma 5.6. Let \( L \in \text{Lat}, P \in \text{Pfin}, p \in P, a, b \in L, f_0, g_0 \in L^P, \theta \in \text{Con}(L^P) \). Assume \( f_0(p) = a, g_0(p) = b \), and \( (f_0, g_0) \in \theta \cup \chi_p \). Then
\[
(f, g) \in \theta \cup \chi_p
\]
for all \( f, g \in L^P \) such that \( f(p) = a, g(p) = b \).
Proposition 5.7. Let $L \in \text{Lat}$, $P \in \mathbb{P}^{\text{fin}}$. Then:

1. for $\theta \in \text{Con}(L^P)$ and $p \in P$,

$$[\Gamma_p'(\theta)](p) = \{ (a, b) \in L \times L \mid (f, g) \in \theta \lor \chi_p \text{ for some } f, g \in L^P \text{ such that } f(p) = a, g(p) = b \}$$

2. $\Gamma_p'$ and $\Delta_p'$ are mutually-inverse order-isomorphisms;

3. $\Gamma_P$ maps $\text{Comp}(L^P)$ onto $(\text{Comp } L)^F$.

Proof. For $a \in L$, let $\bar{a} \in L^P$ denote the constant map with value $a$.

By Lemma 6, $\Gamma' := \Gamma_p'$ is well-defined, as is $\Delta' := \Delta_p'$; (1) also holds. Both $\Gamma'$ and $\Delta'$ are order-preserving.

Let $\theta \in \text{Con}(L^P)$. Let $f, g \in L^P$. Then

$$(f, g) \in (\Delta' \circ \Gamma')(\theta) \iff (f(p), g(p)) \in [\Gamma_p'(\theta)](p) \text{ for all } p \in P$$

$$\iff (f, g) \in \theta \lor \chi_p \text{ for all } p \in P$$

$$\iff (f, g) \in \bigwedge_{p \in P} (\theta \lor \chi_p)$$

$$\iff (f, g) \in \theta \lor \bigwedge_{p \in P} \chi_p$$

$$\iff (f, g) \in \theta.$$

Thus $(\Delta' \circ \Gamma')(\theta) = \theta$, so that $\Delta' \circ \Gamma' = \text{id}_{\text{Con}(L^P)}$.

Let $F \in (\text{Con } L)^F$, $a, b \in L$, $p_0 \in P$. First assume $(a, b) \in F(p_0)$. Then for all $p \in P$, $(m_P(a, b, p_0))(p) \in F(p)$, so that

$$(m_P(a, b, p_0), m_P(b, a, p_0)) \in \Delta'(F)$$

and hence

$$(a, b) \in [(\Gamma' \circ \Delta')(F)](p_0).$$

Therefore $F \leq (\Gamma' \circ \Delta')(F)$.

Now assume $(a, b) \in [(\Gamma' \circ \Delta')(F)](p_0)$. Then

$$(f, g) \in \Delta'(F) \lor \chi_{p_0}$$

for all $f, g \in L^P$ such that $f(p_0) = a, g(p_0) = b$. Hence

$$(\bar{a}, \bar{b}) \in \Delta'(F) \lor \chi_{p_0}.$$
Thus for some \( n \geq 1 \), there exist \( f_1, \ldots, f_n \in L^P \) such that \( \bar{a} = f_1, \bar{b} = f_n \), and for \( 1 \leq i \leq n \),

\[
(f_i, f_{i+1}) \in \begin{cases} 
\Delta'(F) & \text{if } i \text{ odd}, \\
\chi_{p_0} & \text{if } i \text{ even}.
\end{cases}
\]

Therefore \( (a, b) \in F(p_0) \). Hence \( (\Gamma' \circ \Delta') F = F \). We see that \( \Gamma' \circ \Delta' = \text{id}_{(\text{Con}_L)\mathcal{P}} \). Thus \( \Gamma' \) and \( \Delta' \) are inverse order-isomorphisms, which is (2).

Statement (3) follows from (2) and Lemma 5.

Lemma 5.8. Let \( L \in \text{Lat} \), \( P, Q \in \text{P}^{\text{fin}} \). Let \( \nu: P \to Q \) be order-preserving. Let \( \overline{\nu}: \mathcal{P} \to \mathcal{Q} \) be defined by \( \overline{\nu}(p) := \nu(p) \) for all \( p \in P \). Define \( \mu: L^Q \to L^P \) by

\[
\mu(f) := f \circ \nu \quad (f \in L^Q).
\]

Define \( \bar{\mu}: (\text{Comp} L)\mathcal{Q} \to (\text{Comp} L)\mathcal{P} \) by

\[
\bar{\mu}(F) := F \circ \bar{\nu} \quad (F \in (\text{Comp} L)\mathcal{Q}).
\]

Then:

(1) \( \Gamma_P \circ \text{Comp}(\mu) \circ \Delta_Q: (\text{Comp} L)\mathcal{Q} \to (\text{Comp} L)\mathcal{P} \)

is a \( \{0\} \)-\( \vee \)-homomorphism and equals \( \bar{\mu} \);

(2) if \( \nu \) is surjective, then \( \text{Comp} \mu \) is injective.

Proof. By Lemma 4.10, \( \mu \) and \( \bar{\mu} \) are \text{Lat}- and \text{Slat}-morphisms, respectively, so

\[
\text{Comp}(\mu): \text{Comp}(L^Q) \to \text{Comp}(L^P)
\]

is defined. By Proposition 7, \( \Gamma_P \circ \text{Comp}(\mu) \circ \Delta_Q \) is an \text{Slat}-morphism. Fix \( a_0, b_0 \in L, q_0 \in Q \). To prove (1), it suffices to show

\[
\left( \Gamma_P \circ \text{Comp}(\mu) \circ \Delta_Q \right)(F) = \bar{\mu}(F)
\]

for \( F \in (\text{Comp} L)\mathcal{Q} \) defined by

\[
F(q) = \begin{cases} 
\vartheta^L(a_0, b_0) & \text{if } q = q_0, \\
\text{0}_{\text{Con} L} & \text{if } q \neq q_0.
\end{cases}
\]

Thus for \( f, g \in L^Q, (f, g) \in \Delta_Q(F) \) if and only if
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(1) \((f(q_0), g(q_0)) \in \vartheta^L(a_0, b_0)\) and
(2) \(f(q) = g(q)\) for all \(q \in Q \setminus \{q_0\}\).

Let
\[
\eta := \{(h, k) \in L^P \times L^P \mid (h(p), k(p)) \in \vartheta^L(a_0, b_0) \text{ for all } p \in \nu^{-1}(q_0) \\
\text{and } h(p) = k(p) \text{ for all } p \in P \setminus \nu^{-1}(q_0)\}
\]

Then \(\eta \in \text{Con}(L^P)\) by Proposition 7, and \((\mu \times \mu)[\Delta_Q(F)] \subseteq \eta\).

Assume \(\theta \in \text{Con}(L^P)\) and \((\mu \times \mu)[\Delta_Q(F)] \subseteq \theta\). Assume \((a, b) \in \vartheta^L(a_0, b_0)\).

Then \((m_Q(a, b, q_0), m_Q(b, a, q_0)) \in \Delta_Q(F)\); therefore
\[
(\mu \times \mu)(m_Q(a, b, q_0), m_Q(b, a, q_0)) \in \theta,
\]
and so \((m_Q(a, b, q_0) \circ \nu, m_Q(h, a, q_0) \circ \nu) \in \theta\), thus \((a, b) \in [\Gamma'_{\nu}(\theta)](p)\) for all \(p \in \nu^{-1}(q_0)\). Hence
\[
\vartheta^L(a_0, b_0) \subseteq [\Gamma'_{\nu}(\theta)](p)
\]
for all \(p \in \nu^{-1}(q_0)\). If \((h, k) \in \eta\) then
\[
(h(p), k(p)) \in [\Gamma'_{\nu}(\theta)](p)
\]
for all \(p \in P\), so \((h, k) \in \theta\). Hence \(\eta \subseteq \theta\). We have shown that
\[
\eta = (\text{Comp } \mu)(\Delta_Q(F)).
\]

Define \(G \in (\text{Comp } L^P)\) as follows: for all \(p \in P\),
\[
G(p) := \begin{cases} 
\vartheta^L(a_0, b_0) & \text{if } p \in \nu^{-1}(q_0), \\
0_{\text{Con } L} & \text{else}.
\end{cases}
\]

It is clear that \(\Delta_{\nu}(G) = \eta\). Moreover \(G = \bar{\mu}(F)\). Thus (1) holds.

Statement (2) follows from (1) and Lemma 4.10 (2).

In [23], p.98 it is stated that \textbf{Lat}-embeddings map to \textbf{Slat}-embeddings under \text{Comp}. One may construct a counterexample by considering the five-element non-distributive modular lattice. The statement does hold for the embeddings with which we are concerned, however.
Lemma 5.9. Let $L \in \text{Lat}$, $P \in \mathcal{P}$. For all $\Pi \in \mathcal{E}_P$, $\text{Comp} \mu_\Pi^L$ is injective.

Proof. Let $\theta^{(1)}, \theta^{(2)} \in \text{Comp}(L^\Pi)$ be such that

$$(\text{Comp} \mu_\Pi^L)(\theta^{(1)}) = (\text{Comp} \mu_\Pi^L)(\theta^{(2)}).$$

For some $n \geq 0$,

$$\theta^{(r)} = \bigvee_{i=1}^n \vartheta^{L^\Pi}(f_i^{(r)}, g_i^{(r)})$$

for some $f_i^{(r)}, g_i^{(r)} \in L^\Pi$ ($i = 1, \ldots, n$ and $r = 1, 2$). By Lemma 3,

$$(\text{Comp} \mu_\Pi^L)(\theta^{(r)}) = \bigvee_{i=1}^n \vartheta^{L^\Pi}(\mu_\Pi^L(f_i^{(r)}), \mu_\Pi^L(g_i^{(r)})) \quad (r = 1, 2).$$

By Lemma 1, there exists a finite subset $S \subseteq L^P$ containing $\mu_\Pi^L(f_i^{(r)}), \mu_\Pi^L(g_i^{(r)})$ ($i = 1, \ldots, n$ and $r = 1, 2$) such that if $K$ is a sublattice of $L^P$ containing $S$, then

$$\bigvee_{i=1}^n \vartheta^K(\mu_\Pi^L(f_i^{(1)}), \mu_\Pi^L(g_i^{(1)})) = \bigvee_{i=1}^n \vartheta^K(\mu_\Pi^L(f_i^{(2)}), \mu_\Pi^L(g_i^{(2)})).$$

As $S$ is finite, by Propositions 4.4 and 4.14 there exists $\Pi' \in \mathcal{E}_P$ such that $\Pi' \leq \Pi$ and $S \subseteq \text{Im} \mu_\Pi^{L^\Pi}$. As $\mu_\Pi^L = \mu_{\Pi'} \circ \mu_{\Pi', \Pi}$ and $\mu_{\Pi'}$ is injective [Lemmas 4.11 (1) and 4.12 (3)],

$$\bigvee_{i=1}^n \vartheta^{L^\Pi}(\mu_{\Pi', \Pi}^{L^\Pi}(f_i^{(1)}), \mu_{\Pi', \Pi}^{L^\Pi}(g_i^{(1)})) = \bigvee_{i=1}^n \vartheta^{L^\Pi}(\mu_{\Pi', \Pi}^{L^\Pi}(f_i^{(2)}), \mu_{\Pi', \Pi}^{L^\Pi}(g_i^{(2)})).$$

Hence $(\text{Comp} \mu_{\Pi', \Pi}^L)(\theta^{(1)}) = (\text{Comp} \mu_{\Pi', \Pi}^L)(\theta^{(2)})$. By Lemma 8, $\text{Comp} \mu_{\Pi', \Pi}^L$ is injective, so $\theta^{(1)} = \theta^{(2)}$.

The following proof utilizes an idea from [23], pp. 98–100. The theorem, proved independently, appears in [15], Theorem 4.

Theorem 5.10. Let $L \in \text{Lat}$, $P \in \mathcal{P}$. Then $\text{Comp}(L^P) \cong (\text{Comp} L)^\Pi$.
Proof. By Proposition 4.14 (2),
\[ \left( L^\Pi, (\mu_{\Pi, \Pi_2}^L : L^{\Pi_2} \to L^\Pi)_{\Pi \in \mathcal{E}_P} \right) \]
is a filtered limit in \text{Lat} of the filtered system
\[ \left( \left( L^\Pi \right)_{\Pi \in \mathcal{E}_P}, \left( \mu_{\Pi_1, \Pi_2}^L : L^{\Pi_2} \to L^{\Pi_1} \right)_{\Pi_1, \Pi_2 \in \mathcal{E}_P} \right). \]
Hence
\[ \left( \left( \text{Comp}(L^\Pi) \right)_{\Pi \in \mathcal{E}_P}, \left( \mu_{\Pi_1, \Pi_2}^{\text{Comp}} : \text{Comp}(L^{\Pi_2}) \to \text{Comp}(L^{\Pi_1}) \right)_{\Pi_1, \Pi_2 \in \mathcal{E}_P} \right) \]
is a filtered system in \text{Slat}. By Lemma 8 (1), for \( \Pi_1, \Pi_2 \in \mathcal{E}_P \) such that \( \Pi_1 \leq \Pi_2 \),
\[ \Gamma_{\Pi_1} \circ \text{Comp}(\mu_{\Pi_1, \Pi_2}^L) \circ \Delta_{\Pi_2} = \mu_{\Pi_1, \Pi_2}^{\text{Comp} L} : (\text{Comp} L)^{\Pi_2} \to (\text{Comp} L)^{\Pi_1}. \]
By Proposition 4.15,
\[ \left( \left( (\text{Comp} L)^\Pi \right)_{\Pi \in \mathcal{E}_P}, \left( \mu_{\Pi_1, \Pi_2}^{\text{Comp} L} : (\text{Comp} L)^{\Pi_2} \to (\text{Comp} L)^{\Pi_1} \right)_{\Pi_1, \Pi_2 \in \mathcal{E}_P} \right) \]
is a filtered system in \text{Slat} with filtered limit
\[ \left( \left( (\text{Comp} L)^\Pi \right)_{\Pi \in \mathcal{E}_P}, (\mu_{\Pi}^{\text{Comp} L} : (\text{Comp} L)^{\Pi} \to (\text{Comp} L)^{\Pi})_{\Pi \in \mathcal{E}_P} \right). \]
For each \( \Pi \in \mathcal{E}_P \), let \( f'_\Pi := \text{Comp}(\mu_{\Pi_1}^L) \circ \Delta_{\Pi_1} : (\text{Comp} L)^{\Pi} \to \text{Comp}(L^\Pi) \). For \( \Pi_1, \Pi_2 \in \mathcal{E}_P \) such that \( \Pi_1 \leq \Pi_2 \),
\[ f_{\Pi_1} \circ \mu_{\Pi_1, \Pi_2}^{\text{Comp} L} = \text{Comp}(\mu_{\Pi_1}^L) \circ \Delta_{\Pi_1} \circ \Gamma_{\Pi_1} \circ \text{Comp}(\mu_{\Pi_1, \Pi_2}^L) \circ \Delta_{\Pi_2} \]
\[ = \text{Comp}(\mu_{\Pi_1}^L) \circ \text{Comp}(\mu_{\Pi_1, \Pi_2}^L) \circ \Delta_{\Pi_2} \]
\[ = \text{Comp}(\mu_{\Pi_1}^L \circ \mu_{\Pi_1, \Pi_2}^L) \circ \Delta_{\Pi_2} \]
\[ = \text{Comp}(\mu_{\Pi_2}^L) \circ \Delta_{\Pi_2} \]
\[ = f'_{\Pi_2} \]
by Lemma 4.12 (3). Hence there exists a unique \text{Slat}-morphism
\[ F : (\text{Comp} L)^{\mathcal{P}} \to \text{Comp}(L^\mathcal{P}) \]
such that
\[ F \circ \mu_{\Pi}^{\text{Comp}L} = \text{Comp}(\mu_{\Pi}^L) \circ \Delta_{\Pi} \]
for all \( \Pi \in \mathcal{E}_P \). By Lemma 4.11, \( \mu_{\Pi}^{\text{Comp}L} \) is injective for all \( \Pi \in \mathcal{E}_P \). If \( f_1, f_2 \in (\text{Comp} L)^\Pi \) and \( F(f_1) = F(f_2) \), then by Proposition 4.15 there exist \( \Pi \in \mathcal{E}_P \) and \( g_1, g_2 \in (\text{Comp} L)^\Pi \) such that
\[ f_i = \mu_{\Pi}^{\text{Comp}L}(g_i) \quad (i = 1, 2). \]
Hence \((\text{Comp}(\mu_{\Pi}^L) \circ \Delta_{\Pi})(g_1) = (\text{Comp}(\mu_{\Pi}^L) \circ \Delta_{\Pi})(g_2)\). By Proposition 7 and Lemma 9, \( g_1 = g_2 \), so that \( f_1 = f_2 \). Therefore \( F \) is injective.

Now assume \( \theta \in \text{Comp}(L^P) \). Then for some \( n \geq 0 \), there exist \( f_1, \ldots, f_n, g_1, \ldots, g_n \in L^P \) such that
\[ \theta = \bigvee_{i=1}^{n} \varrho_{L^P}(f_i, g_i). \]
By Proposition 4.14, there exists \( \Pi \in \mathcal{E}_P \) such that \( f_i, g_i \in \text{Im} \mu_{\Pi}^L \) \((i = 1, \ldots, n)\). Let \( h_i, k_i \in L^\Pi \) be such that \( f_i = \mu_{\Pi}^L(h_i), g_i = \mu_{\Pi}^L(k_i) \) \((i = 1, \ldots, n)\). Then
\[ \bigvee_{i=1}^{n} \varrho_{L^\Pi}(h_i, k_i) \in \text{Comp}(L^\Pi) \]
and by Lemma 3
\[ (\text{Comp}\mu_{\Pi}^L)\left( \bigvee_{i=1}^{n} \varrho_{L^\Pi}(h_i, k_i) \right) = \bigvee_{i=1}^{n} \varrho_{L^P}(f_i, g_i) = \theta \]
so that \( F \) is surjective. Hence \( F \) is an isomorphism.

6. The congruence lattice of a Priestley power of a lattice

In this section we determine the structure of the congruence lattice of a Priestley power of a lattice in terms of the lattice and the Priestley space (Theorem 7 and Corollaries 8, 10, and 11). We derive as corollaries the known results that, when the lattice or the space is finite, the problem of \S 1 has a positive solution (Corollaries 12 and 13).

In \S 5 we determined the structure of the distributive semilattice of compact congruences of a Priestley power of a lattice. To go from this semilattice to the congruence lattice, we use Stone duality.
**Lemma 6.1.** Every trivially ordered Priestley space is a Stone space.

**Proof.** Consider a trivially ordered Priestley space. It is homeomorphic to \( P(B) \) for some Boolean algebra \( B \). This space has a basis consisting of the sets
\[
\{ F \in P(B) \mid a \in F \} \quad (a \in B).
\]
The map
\[
F \mapsto B \setminus F \quad [F \in P(B)]
\]
is a bijection from \( P(B) \) to \( S(B) \), which has a basis consisting of the sets
\[
\{ I \in B^\sigma \mid I \text{ prime and } a \not\in I \} \quad (a \in B),
\]
so that the map is a homeomorphism.

**Lemma 6.2** ([29], Lemma 6). The product of sober spaces is sober.

**Lemma 6.3.** Let \( X \) and \( Y \) be Stone spaces. Then \( X \times Y \) is a Stone space with basis \( \{ U \times V \mid U \in \mathcal{C}O(X), V \in \mathcal{C}O(Y) \} \subseteq \mathcal{C}O(X \times Y) \).

**Proof.** Obviously \( X \times Y \) is \( T_0 \) and has basis
\[
\{ U \times V \mid U \in \mathcal{C}O(X), V \in \mathcal{C}O(Y) \} \subseteq \mathcal{C}O(X \times Y).
\]
By Lemma 2, it is sober.

**Lemma 6.4.** Let \( X \) and \( Y \) be Stone spaces and \( J \) a set. For all \( j \in J \), let \( S_j \in \mathcal{C}O(X) \) and \( T_j \in \mathcal{C}O(Y) \). Let \( R := \bigcup_{j \in J} (S_j \times T_j) \in \mathcal{O}(X \times Y) \). Then:

1. for all \( y \in Y \),
\[
\bigcup\{ U \in \mathcal{C}O(X) \mid U \times \{ y \} \subseteq R \} = \bigcup\{ S_j \mid j \in J \text{ and } y \in T_j \};
\]
2. for all \( W \in \mathcal{C}O(X) \),
\[
\left\{ y \in Y \mid W \subseteq \bigcup\left\{ U \in \mathcal{C}O(X) \mid U \times \{ y \} \subseteq R \right\} \right\} \in \mathcal{O}(Y);
\]
3. for all \( y_0 \in Y \),
\[
\bigcap\{ T_j \mid j \in J \text{ and } y_0 \in T_j \} \cap \bigcap\{ Y \setminus T_j \mid j \in J \text{ and } y_0 \not\in T_j \}
\]
is a subset of the set of all \( y \in Y \) such that
\[
\bigcup\{ U_1 \in \mathcal{C}O(X) \mid U_1 \times \{ y_0 \} \subseteq R \} = \bigcup\{ U_2 \in \mathcal{C}O(X) \mid U_2 \times \{ y \} \subseteq R \}.
\]
Proof. (1) Fix \( y \in Y \). Let \( U \in \mathcal{CO}(X) \) be such that \( U \times \{ y \} \subseteq R \). Then for all \( u \in U \) there exists \( j_u \in J \) such that

\[
(u, y) \in S_{j_u} \times T_{j_u}.
\]

Hence \( u \in \{ S_j \mid j \in J \text{ and } y \in T_j \} \).

Now assume \( j \in J, s \in S_j, \text{ and } y \in T_j \). Then \( S_j \in \mathcal{CO}(X) \) is such that \( S_j \times \{ y \} \subseteq R \).

(2) Let \( W \in \mathcal{CO}(X) \) and \( y_0 \in Y \) be such that

\[
W \subseteq \bigcup \{ U \in \mathcal{CO}(X) \mid U \times \{ y_0 \} \subseteq R \}.
\]

By (1), for some \( n \geq 0 \) there exist \( j_1, \ldots, j_n \in J \) such that

\[
W \subseteq \bigcup_{k=1}^{n} S_{j_k} \quad \text{and} \quad y_0 \in \bigcap_{k=1}^{n} T_{j_k} =: T.
\]

For any \( t \in T \),

\[
W \subseteq \bigcup_{k=1}^{n} S_{j_k} \subseteq \bigcup \{ U \in \mathcal{CO}(X) \mid U \times \{ t \} \subseteq R \}.
\]

As \( T \in \mathcal{O}(Y) \), (2) follows.

(3) Assume

\[
y \in \bigcap \{ T_j \mid j \in J \text{ and } y_0 \in T_j \} \cap \bigcap \{ Y \setminus T_j \mid j \in J \text{ and } y_0 \notin T_j \}.
\]

By (1),

\[
\bigcup \{ U_1 \in \mathcal{CO}(X) \mid U_1 \times \{ y_0 \} \subseteq R \}
\]

equals

\[
\bigcup \{ S_j \mid j \in J \text{ and } y_0 \in T_j \} = \bigcup \{ S_j \mid j \in J \text{ and } y \in T_j \}
= \bigcup \{ U_2 \in \mathcal{CO}(X) \mid U_2 \times \{ y \} \subseteq R \}.
\]

The next lemma is simple.
Lemma 6.5. Let \( X \) be a Stone space. Then:
(1) \( \mathcal{O}(X) \) is an algebraic lattice;
(2) \( \mathcal{C}\mathcal{O}(X) = \kappa\mathcal{O}(X) \).

After proving the next proposition, the author noted that the first part follows from [12], Theorem II.4.10.

Proposition 6.6. Let \( X \) be a Stone space and \( Y \) a trivially ordered Priestley space. Define a map
\[
\Psi: \mathcal{O}(X \times Y) \rightarrow \mathcal{O}(X)^Y
\]
as follows: for all \( R \in \mathcal{O}(X \times Y) \) and \( y \in Y \), let
\[
[\Psi(R) ][y] := \bigcup \{ U \in \mathcal{C}\mathcal{O}(X) \mid U \times \{ y \} \subseteq R \}.
\]

Then \( \Psi \) is an order-isomorphism. The restriction of \( \Psi \) to \( \mathcal{C}\mathcal{O}(X \times Y) \) maps onto \( \mathcal{C}\mathcal{O}(X)^Y \).

Proof. By Lemmas 1 and 3, every \( R \in \mathcal{O}(X \times Y) \) equals \( \bigcup (S_j \times T_j) \) for some set \( J \) and \( S_j \in \mathcal{C}\mathcal{O}(X) \), \( T_j \in \mathcal{C}\mathcal{O}(Y) \) (\( j \in J \)). (If \( R \in \mathcal{C}\mathcal{O}(X \times Y) \), we may assume \( J \) is finite, so that, by Lemma 4 (1) and (3), \( \Psi(R) \in \mathcal{C}\mathcal{O}(X)^Y \).) By Lemma 4 (2), \( \Psi \) is well-defined. It is clearly order-preserving.

Assume \( R, S \in \mathcal{O}(X \times Y) \) and \( \Psi(R) \leq \Psi(S) \). Assume \((x, y) \in R \). Then there exists \( U \in \mathcal{C}\mathcal{O}(X) \) such that \( x \in U \) and \( U \times \{ y \} \subseteq R \). Hence
\[
U \subseteq [\Psi(R) ][y] \subseteq [\Psi(S) ][y].
\]
Therefore there exists \( U_0 \in \mathcal{C}\mathcal{O}(X) \) such that \( x \in U_0 \) and \( U_0 \times \{ y \} \subseteq S \). Hence \((x, y) \in S \). Therefore \( R \subseteq S \) and \( \Psi \) is an order-embedding.

Now assume \( f \in \mathcal{O}(X)^Y \). Suppose \( U \in \mathcal{C}\mathcal{O}(X) \), \( y \in Y \), and \( U \subseteq f(y) \). Then there exists \( T_{U,y} \in \mathcal{C}\mathcal{O}(Y) \) such that \( y \in T_{U,y} \) and \( U \subseteq f(t) \) for all \( t \in T_{U,y} \). [If \( f \in \mathcal{C}\mathcal{O}(X)^Y \), let \( T_{U,y} := f^{-1}(f(y)) \).]

Let
\[
R := \bigcup_{y \in Y} \bigcup_{U \in \mathcal{C}\mathcal{O}(X)} (U \times T_{U,y}) \in \mathcal{O}(X \times Y).
\]
[If \( f \in \mathcal{C}\mathcal{O}(X)^Y \) and \( T_{U,y} = f^{-1}(f(y)) \) \((U \in \mathcal{C}\mathcal{O}(X) \), \( y \in Y \) such that \( U \subseteq f(y) \))], this set equals \( \bigcup_{y \in Y} \big( f(y) \times f^{-1}(f(y)) \big) \), which may be reduced to a finite union.
since \( \text{Im} f \) is finite, so belongs to \( \mathcal{CO}(X \times Y) \).] By Lemma 4 (1), for all \( y \in Y \),

\[
[\Psi(R)](y_0) = \bigcup \{ U \in \mathcal{CO}(X) \mid U \subseteq f(y) \}
\]

for some \( y \in Y \) and \( y_0 \in T_{U,y} \}

which equals \( f(y_0) \). Hence \( \Psi(R) = f \), so \( \Psi \) is surjective. Therefore \( \Psi \) is an order-isomorphism.

\[\blacklozenge\]

**Theorem 6.7.** Let \( L \in \text{Lat}, P \in \mathbf{P} \). Then \( \text{Con}(L^P) \cong (\text{Con} L)^\mathbf{P} \).

**Proof.** As \( \text{Comp} L \in \text{DSlat} \), there exists a Stone space \( X \) such that \( \mathcal{CO}(X) \cong \text{Comp} L \).

By Proposition 6, \( (\text{Comp} L)^\mathbf{P} \cong \mathcal{CO}(X \times \mathbf{P}) \), where \( X \times \mathbf{P} \) is a Stone space by Lemmas 1 and 3. By Theorem 5.10, \([\text{Comp} L]^\mathbf{P}|^\sigma \cong \text{Con}(L^P)\). By Lemma 5, \( \mathcal{CO}(X \times \mathbf{P})|^\sigma \cong \mathcal{O}(X \times \mathbf{P}) \). By Proposition 6, \( \mathcal{O}(X \times \mathbf{P}) \cong \mathcal{O}(X)^\mathbf{P} \). By Lemma 5 again, \( \mathcal{O}(X)^\mathbf{P} \cong [(\text{Comp} L)^\mathbf{P}]^\sigma \cong (\text{Con} L)^\mathbf{P} \). Hence \( \text{Con}(L^P) \cong (\text{Con} L)^\mathbf{P} \). \[\blacklozenge\]

From Corollary 3.7, we get the following.

**Corollary 6.8.** Let \( L \in \text{Lat}, M \in \mathbf{D} \). Then

\[
\text{Con}(L^{P(M)}) \cong \text{Slat}\left(\left(\text{Comp} L, \lor, 0_{\text{Con} L}\right), \left(\mathcal{M}_{\text{Bool}}\right), \cap, 1_{\mathcal{M}_{\text{Bool}}}\right)\).
\]

**Lemma 6.9.** Let \( M \in \mathbf{D} \). Then \( (\mathcal{M}_{\text{Bool}})^\sigma \cong \text{Con} M \).

**Proof.** Because \( P(\mathcal{M}_{\text{Bool}}) \) is trivially ordered, we have

\[
(\mathcal{M}_{\text{Bool}})^\sigma \cong U\left(P(\mathcal{M}_{\text{Bool}})\right) \cong \mathcal{O}\left(P(\mathcal{M}_{\text{Bool}})\right) \cong \mathcal{O}\left(P(M)\right) \cong \text{Con} M.
\]

**Corollary 6.10.** Let \( L \in \text{Lat}, M \in \mathbf{D} \). Then

\[
\text{Con}(L^{P(M)}) \cong \text{Slat}\left(\left(\text{Comp} L, \lor, 0_{\text{Con} L}\right), \left(\text{Con} M, \cap, 1_{\text{Con} M}\right)\right).
\]

The next corollary follows from Corollary 3.7.
Corollary 6.11. Let $L \in \text{Lat}$, $M \in D$. Then
$$\text{Con}(L^{P(M)}) \cong (\text{Con} L)^{P(\text{Con} M)}.$$ 

Corollary 6.12 ([9], Theorem 2.1). Let $L$ be a lattice and $P$ a finite poset with $n$ elements. Then $\text{Con}(L^P) \cong (\text{Con} L)^n$.

Proof. As $P$ is a discrete space, $(\text{Con} L)^P = (\text{Con} L)^{\text{Con} M}$ by Corollary 3.7. The result follows from Theorem 7.

Corollary 6.13 ([26], Theorem). Let $L$ be a finite lattice, $M \in D$. Then $\text{Con}(L^{P(M)}) \cong (\text{Con} L)^{P(\text{Con} M)}$.

Proof. As $\text{Comp} L$ is finite,
$$\text{Slat} \left( (\text{Comp} L, \lor, 0_{\text{Con} L}), (\text{Con} M, \cap, 1_{\text{Con} M}) \right) \cong \text{Slat} \left( (\text{Comp} L, \lor, 0_{\text{Con} L}), (\text{Con} M, \cap, 1_{\text{Con} M}) \right) \cong \text{Slat} \left( (\text{Comp} L, \lor, 0_{\text{Con} L}), (\text{Con} M, \cap, 1_{\text{Con} M}) \right).$$

By Corollary 10, the left-hand side is isomorphic to $\text{Con}(L^{P(M)})$. By Corollary 3.7, the right-hand side is isomorphic to $(\text{Con} L)^{P(\text{Con} M)}$.

7. A counterexample

In this section we show that the answer to Schmidt’s question ($\S 1$) is in general negative. As stated in $\S 1$, Grätzer and Schmidt have determined exactly when it has a positive solution ([15], Theorem 3); our results were obtained independently.

Lemma 7.1. Let $S$ be a chain with 0. Let $T \in D$. Then
$$\text{Slat} \left( (S, \lor, 0_S, \text{Con} L), (T^\sigma, \cap, T) \right) = \{ f \in (T^\sigma)^S \mid f(0_S) = T \}.$$ 

Proof. Let $f \in \text{Slat} \left( (S, \lor, 0_S, \text{Con} L), (T^\sigma, \cap, T) \right)$. Assume $s_1$, $s_2 \in S$ and $s_1 \leq s_2$. Then $f(s_2) = f(s_1 \lor s_2) = f(s_1) \cap f(s_2)$, so that $f(s_2) \subseteq f(s_1)$.

Now assume $f \in (T^\sigma)^S$. Let $s_1$, $s_2 \in S$. Without loss of generality $s_1 \leq s_2$. Hence $f(s_1 \lor s_2) = f(s_2)$. As $f(s_2) \subseteq f(s_1)$, we have $f(s_1) \cap f(s_2) = f(s_2)$. Thus $f(s_1 \lor s_2) = f(s_1) \cap f(s_2)$.

Corollary 7.2. Let $C$ be a chain. Let $S := 1 \oplus (C^o)$, $T \in D$. Then:
(1) $\text{Slat}((S, \vee, 0_S), (T^\sigma, \cap, T)) \cong (T^\sigma)^C$;
(2) $\text{Slat}^\text{fin}((S, \vee, 0_S), (T^\sigma, \cap, T)) \cong \{ f \in (T^\sigma)^C \mid \text{Im } f \text{ finite} \}$.

Lemma 7.3. The poset $\{ f \in [\mathcal{P}(\mathbb{N})^\sigma]^\mathbb{N} \mid \text{Im } f \text{ finite} \}$ is not a complete lattice.

Proof. For all $n_0 \in \mathbb{N}$, define the map
$$f_{n_0} : \mathbb{N} \to \mathcal{P}(\mathbb{N})^\sigma$$
as follows. For all $n \in \mathbb{N}$,
$$f_{n_0}(n) := \begin{cases} \{ \emptyset, \{n_0\} \} & \text{if } n \geq n_0, \\ \{ \emptyset \} & \text{if } n < n_0. \end{cases}$$

Then for all $n_0 \in \mathbb{N}$,
$$f_{n_0} \in P := \{ f \in [\mathcal{P}(\mathbb{N})^\sigma]^\mathbb{N} \mid \text{Im } f \text{ finite} \}.$$

For $n \in \mathbb{N}$,
$$\left( \bigvee_{\mathcal{P}(\mathbb{N})^\sigma} \{ f_{n_0} \mid n_0 \in \mathbb{N} \} \right)(n) = \mathcal{P}(\{1, \ldots, n\}).$$

Suppose for a contradiction that
$$g := \bigvee_P \{ f_{n_0} \mid n_0 \in \mathbb{N} \}$$
exists. Then there exists $k_0 \in \mathbb{N}$ such that $k_0 \leq n$ implies $g(k_0) = g(n)$ ($n \in \mathbb{N}$). For all $n \in \mathbb{N}$, $\mathcal{P}(\{1, \ldots, n\}) \subseteq g(n)$; if $n \geq k_0$, then $\mathcal{P}(\{1, \ldots, n\}) \subseteq g(k_0)$.

Define $h : \mathbb{N} \to \mathcal{P}(\mathbb{N})^\sigma$ as follows: for all $n \in \mathbb{N}$,
$$h(n) := \begin{cases} \mathcal{P}(\{1, \ldots, n\}) & \text{if } n \leq k_0, \\ g(k_0) & \text{if } k_0 < n. \end{cases}$$

Then $h \in P$ and for all $n_0 \in \mathbb{N}$,
$$f_{n_0} \leq h.$$

Hence $g \leq h$; but $g(k_0)$ is infinite and $h(k_0)$ is finite, a contradiction. 


Proposition 7.4. There exist $L \in \text{Lat}$ and $M \in \text{D}$ such that

$$\text{Con}(L^{P(M)}) \not\sim (\text{Con}L)^{P(\text{Con}M)}.$$  

Proof. Let $M := \mathcal{P}(N)$. Note that $1 \oplus (N^0) = \kappa \left( 1 \oplus (N^0) \right)$. It is well-known that there exists $L \in \text{Lat}$ such that $\text{Con}L \cong 1 \oplus (N^0)$ (see, for example, [27], Theorem). Hence $\text{Comp}L \cong 1 \oplus (N^0)$.

By Lemma 6.9,

$$(\text{Con}L)^{P(\text{Con}M)} \cong (\text{Con}L)^{P(M^\sigma)}.$$  

By Corollary 3.7,

$$(\text{Con}L)^{P(M^\sigma)} \cong S\text{lat}^{\text{fin}} \left( \left( \text{Comp}L, \lor, 0_{\text{Con}L}, (M^\sigma, \cap, M) \right) \right)$$

$$\cong S\text{lat}^{\text{fin}} \left( \left( 1 \oplus (N^0), \lor, 0, \left( \mathcal{P}(N)^\sigma, \cap, \mathcal{P}(N) \right) \right) \right).$$  

By Corollary 2 (2) we have

$$(\text{Con}L)^{P(\text{Con}M)} \cong \{ f \in [\mathcal{P}(N)^\sigma]^N \mid \text{Im } f \text{ finite } \},$$  

which is not a complete lattice by Lemma 3, so cannot be isomorphic to a congruence lattice.

References


J. D. Farley, Mathematical Institute, University of Oxford, 24–29 St. Giles', Oxford OX1 3LB, United Kingdom; current affiliation: Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA; e-mail: farley@msri.org